LECTURE NOTES

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OPERATIONS RESEARCH (MMAF183T20)

UNIT-I

Preview

- Gomory's All Integer Cutting Plane Method
- Gomory's mixed Integer Cutting Plane method
- Branch and Bound Method.
- Dynamic Programming Terminology –Optimal Decision Policy

Introduction:-

In linear programming problem, the variables are allowed to take any real or fractional value, however in the real life problem the fractional values of variables has *no significance*, for example it does not make sense that *3.5 workers* are required for the project, *2.4 machines* required for the work shop ect.,

The integer solution is obtained by *rounding off* the optimal value of the variables to the *nearest integer*. This approach is very easy, however it may not satisfies all the given constrains and also the value of the objective function so obtained *may not be optimal.*

This type of difficulties *can be avoided* if the given problem is solved by integer programming method. The integer linear programming problems are those in which some or all the variables are restricted to integer value (discrete value).

Application:-

Capital Budgeting, construction scheduling, plant location and size, routing and shifting schedule, batch size , capacity extension fixed charges, ect., are few problems that demonstrate the area of application of integer programming problem.

Types of Integer Programming Problems

Integer programming problems can be classified in to three categories

(i) ***Pure (all) integer programming problems*** in which all decision variables are restricted to integer values.

(ii) Mixed integer programming problems in which some, but not all, of the decision variables restricted to integer values.

(iii) Zero-one integer programming problems in which all decision variables are restricted to either zero (0) or one (1).

Standard form of Pure Integer Programming Problem

The pure integer programming problem in its standard form can be written as follows

Max $Z=c_1x_1+c_2x_2+....c_nx_n$,

Subject to the constraints

$a_{11}x_1+a_{12}x_2+...............a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+...............a_{2n}x_n=b_2$

$a_{31}x_1+a_{32}x_2+...............a_{3n}x_n=b_3$

;

$a_{m1}x_1+a_{m2}x_2+...............a_{mn}x_n=b_m$

and $x_1, x_2, x_3, ...x_n \geq 0$ and are integers.

Cut

A cut is a linear constraint added to the given LP Problem, it is also called additional linear constraint.(or fractional cut)

Enumeration and Cutting Plane Solution Concept

- The cutting plane method is used to solve integer linear programming problem
- The cutting plane method was developed by R.E. Gomory in the year 1956.
- The cutting plane method is based on the generation of sequence of linear inequality called cut.
- The hyper plane boundary of a cut is called cutting plane.

Example:-

Consider the following linear integer programming problem

Max $Z=14x_1+16x_2$

Subject to the constraints

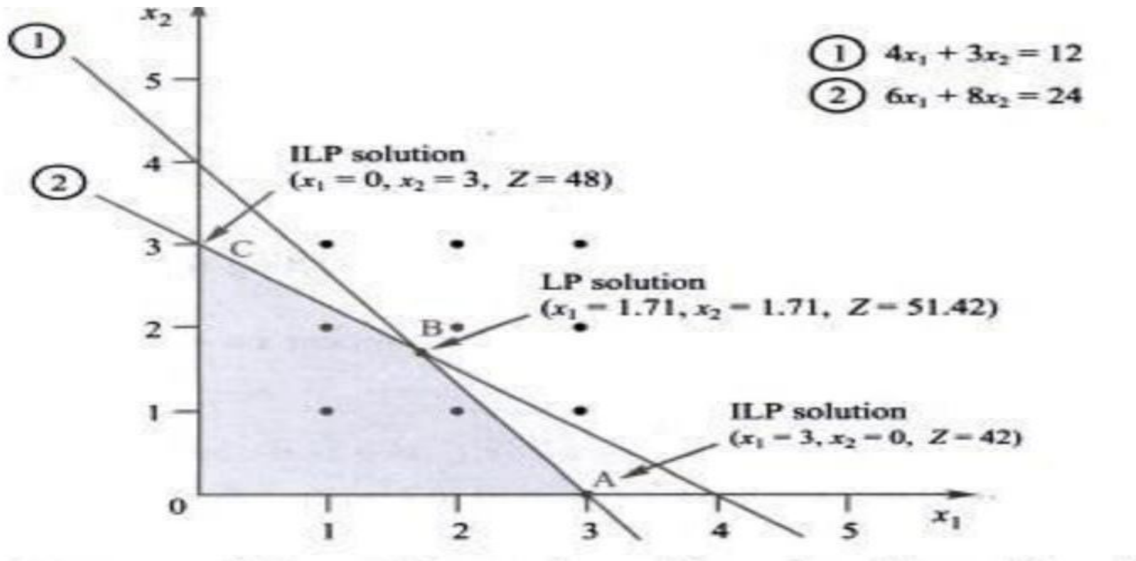
$4x_1+3x_2 \leq 12$ and $6x_1+8x_2 \leq 24$

and $x_1, x_2 \geq 0$ and are integers.

Solution:-

Relax the integer requirements and apply graphical method,

Graphical Method



The feasible region is OABC.

We get the following solution x_1=1.71 and x_2=1.71 , Max Z=51.42.

This solution does not satisfies the integer requirements of the variables x_1 and x_2.

Rounding off the solution, we get x_1=2 and x_2=2, Max Z=60, But it is not feasible solution, since it does not satisfies the given inequalities

$4x_1+3x_2= 4(2)+3(2)=14$ is not less than 12

$6x_1+8x_2=6(2)+8(2)=28$ is not less than 24,

Therefore the solution is infeasible.

The dots in the graph represents the possible integer solution lies in the feasible region of the LP problem. The dots are also called lattice points.

Cutting plane Method

The optimal lattice point C is obtained by cutting away the small portion above the dotted line. This suggest a solution procedure that successively cuts down (reduce) the feasible solution space until the integer valued corner is found.

The optimal integer solution to the given LP problem is x_1=0, x_2=3 and Max Z=48.

Remarks:-

- Reducing the feasible region by adding the extra constraints (cut) can never give an improved objective function value.
- If Z_{IP} is the minimum value of the objective function in an ILP problem Z_{LP} is the minimum value of the objective function in an LP problem, then $Z_{IP} \geq Z_{LP}$.

Gomory's all integer cutting plane method

A systematic procedure that generates cuts (additional linear constraints) so as to ensure the integer solution to the LP problem in a finite number of steps is called Gomory all integer cutting plane algorithm.

Properties of Gomory's Algorithm

- Additional linear constraints never cut off that portion of the original feasible solution space that contains a feasible integer solution to the original problem.
- Each new additional constraints (hyper planes) cut off the current non integer optimal solution to the linear programming problem.

Construction of additional constraints (CUT)

In the optimal simplex table, we select one of the row, called source row for which basic variables is non integer. The desired cut is developed by considering only fractional part of the coefficient in the source row, for this reason such a cut is also called fractional cut.

Procedure of Gomory's Algorithm

Step:-01

Solve the given LP problem using simplex method by ignoring integer requirements.

Step:-02

If all the variables in the X_B column of the final simplex table is non negative integer, then the current solution is optimal. otherwise go to next step.

Step:-03

If some of the basic variables do not have non negative integer value, then an additional linear constraints called Gomory's linear constraints (cut) is generated, after having generated linear constraints (cutting plane) , it is added to the bottom of the optimum simple table so that the solution no longer remains feasible.

Step:-04

The new problem is then solved by dual simplex method, If the optimum solution , so obtained is integer. Which is the required one, otherwise repeat Step: 03 until all basic variables assume non negative integer values.

Problem:-01

Solve the following integer programming problem using Gomory's Cutting plane method, Max $Z=x_1+x_2$, subject to the conditions $3x_1+2x_2 \leq 5$, $x_2 \leq 2$ and $x_1, x_2 \geq 0$ and integer.

Solution:-

Step:-01

Let us find the optimal solution by simplex method *(ignoring integer conditions)*

Simple method

First we have to convert the inequality constraints to equality constraints by adding slack or surplus variables.

$3x_1+2x_2 \leq 5$, Add slack variable $s_1 \geq 0$, we have $3x_1+2x_2+s_1=5$,

$x_2 \leq 2$, Add slack variable $s_2 \geq 0$, we have $x_2+s_2=2$,

We have to add the slack variables in the objective function with zero coefficients.

Max $Z=x_1+x_2+0s_1+0s_2$

Initial Basic feasible solution

Let $x_1=0$ and $x_2=0$ (non Basic variables) substitute in the given problem, we have $s_1=5$, $s_2=2$ (basic variables) and Max $z=0$.

Initial Simple table

	C_j	1	1	0	0		
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios	
0	$s_1=5$	[3]	2	1	0	5/3	←
0	$s_2=2$	0	1	0	1	---	
z=0	Z_j	0	0	0	0		
	Z_j-C_j	-1	-1	0	0		
		↑					

Arrows represents selected row and column. Square bracket [ ] represent the diagonal element.(x_1 enters the basis and s_1 leaves the basis)

Simplex iteration Rule:

(i) Divide the entire selected row by its diagonal element

(ii) All selection column entries zero except diagonal entry

(ii) Remaining all the position values using the formula

present position value - (corresponding selected row value) x (corresponding selected column value) / (diagonal element)

First row values for the new table

$$\frac{5}{3} \quad \frac{3}{3}=1 \quad \frac{2}{3} \quad \frac{1}{3} \quad \frac{0}{3}=0$$

Second row values for the new table

$$2-\frac{(5)(0)}{3}=2, \quad 0-\frac{(3)(0)}{3}=0, \quad 1-\frac{(2)(0)}{3}=1, \quad 0-\frac{(1)(0)}{3}=0, \quad 1-\frac{(0)(0)}{3}=1$$

First simplex table

	C_j	1	1	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
1	$x_1=5/3$	1	2/3	1/3	0	5/2
0	$s_2=2$	0	[1]	0	1	2
z=5/3	Z_j	1	2/3	1/3	0	
	Z_j-C_j	0	-1/3	1/3	0	

Max $Z=5/3+0+0s_1+0s_2=5/3$

x_2 enters the basis and s_2 leaves the basis, the diagonal element is 1

First row values for the new table

$$\frac{5}{3}-\frac{(2)(2/3)}{1}=\frac{1}{3} \quad 1-\frac{(0)(2/3)}{1}=1 \quad \frac{2}{3}-\frac{(2/3)(1)}{1}=0 \quad \frac{1}{3}-\frac{(2/3)(0)}{3}=\frac{1}{3} \quad 0-\frac{(2/3)(1)}{1}=\frac{-2}{3}$$

Second row values for the new table

$$\frac{2}{1}=2 \quad \frac{0}{1}=0 \quad \frac{1}{1}=1 \quad \frac{0}{1}=0 \quad \frac{1}{1}=1$$

Second simplex table

	C_j	1	1	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
1	$x_1=1/3$	1	0	1/3	-2/3	
1	$x_2=2$	0	1	0	1	
z=7/2	Z_j	1	1	1/3	1/3	
	Z_j-C_j	0	0	1/3	1/3	

Max $Z=1/3+2+0s_1+0s_2=7/2$

Since $Z_j-C_j \geq 0$ for all j=1,2,3,4, therefore by the simplex procedure the above table is optimum, and the optimum solution is given by $x_1=1/3$, $x_2=2$ and Max z=7/2.

Step-02

In the current optimal solution all the basic variables in the basis are not integer.

Therefore the above solution is not desirable.

Step:-03

Since x_1 is the only basis variable whose value is *non-negative fraction*, so we consider the first row as source row for generating *Gomory fractional cut*.

[Rule:- Fractional cut

Each non integer coefficient is factorized into integer and fractional part in such a way that fractional part is positive.]

$0+1/3 = x_1+0x_2+(1/3)s_1-(2/3)s_2$

$0+1/3 = (1+0)\ x_1+(0+0)\ x_2+(0+1/3)s_1+(-1+1/3)s_2$

Therefore the Gomory's fractional cut -I is written as follows

$(1/3) \leq (1/3)\ s_1+(1/3)s_2$

$(1/3)= (1/3)\ s_1+(1/3)s_2 - s_{g1}$ (ADD surplus variable S_{g1})

$(-1/3)= (-1/3)\ s_1-(1/3)s_2 + s_{g1}$

(Multiply by -1, so that coefficient of s_{g1} is positive)

Let us add the above equation at the end of the optimum table and apply dual simplex method.

First Dual simplex Table

	C_j	1	1	0	0	0
CB_j	Basic variables	x_1	x_2	s_1	s_2	s_{g1}
1	$x_1=1/3$	1	0	1/3	-2/3	0
1	$x_2=2$	0	1	0	1	0
0	$s_{g1}=-1/3$	0	0	[-1/3]	-1/3	1
z=7/2	Z_j	1	1	1/3	1/3	0
	$C_j - Z_j$	0	0	-1/3	-1/3	0
	Ratio	-	-	1	1	-

Ratio=$(C_j-Z_j)/y_{3j}$, calculate for all $y_3j<0$

Since the values of the ratio row are positive , therefore the above table is not optimum.

(Rule:-Suppose if all are negative , it is optimal)

Always we have to select the row that contain Gomory's slack variable and column with least positive ratio value.

s_{g1} leaves the basis, s_1 enters the basis, diagonal element is -1/3.

First row of new table

$$1/3-\frac{(-1/3)(1/3)}{(-1/3)}=0 \quad 1-\frac{(0)(1/3)}{(-1/3)}=1 \quad 1/3-\frac{(-1/3)(1/3)}{(-1/3)}=0 \quad -2/3-\frac{(-1/3)(1/3)}{(-1/3)}=-1$$

$$0-\frac{(1/3)(1)}{(-1/3)}=1$$

Second row of new table

$$2-\frac{(0)(1/3)}{(-1/3)}=2 \quad 0-\frac{(0)(0)}{(-1/3)}=0 \quad 1-\frac{(0)(0)}{(-1/3)}=1 \quad 1-\frac{(0)(-1/3)}{(-1/3)}=1 \quad 0-\frac{(0)(1)}{(-1/3)}=0$$

Last row of the new table (divide the entire row by -1/3)

$$\frac{(-1/3)}{(-1/3)}=1 \quad \frac{0}{(-1/3)}=0 \quad \frac{0}{(-1/3)}=0 \quad \frac{(-1/3)}{(-1/3)}=1 \quad \frac{(-1/3)}{(-1/3)}=1 \quad \frac{1}{(-1/3)}=-3$$

Second Gomory's cut table

	C_j	1	1	0	0	0
CB_j	Basic variables	x_1	x_2	s_1	s_2	S_{g1}
1	$x_1=0$	1	0	0	-1	1
1	$x_2=2$	0	1	0	1	0
0	$s_1=1$	0	0	1	1	-3
z=2	Z_j	1	1	0	0	1
	$C_j - Z_j$	0	0	0	0	-1

Since $C_j - Z_j \geq 0$ for all j=1,2,3,4,5,6. Therefore the above table is optimum.

Hence the optimum integer solution is $x_1=0$, $x_2=2$ and Max Z=2.

Problem:-02

Solve the following integer programming problem using Gomory's Cutting plane method, Max $Z=5x_1+7x_2$, subject to conditions $-2x_1+3x_2 \leq 6$, $6x_1+x_2 \leq 30$ and $x_1, x_2 \geq 0$ and integer.

Solution:-

Step:-01

Let us find the optimal solution by simplex method *(ignoring integer conditions)*

Simple method

First we have to convert the inequality constraints to equality constraints by adding slack or surplus variables.

$-2x_1+3x_2 \leq 6$, Add slack variable $s_1 \geq 0$, we have $-2x_1+3x_2+s_1=6$,

$6x_1+x_2 \leq 30$, Add slack variable $s_2 \geq 0$, we have $6x_1+x_2+s_2=30$,

We have to add the slack variables in the objective function with zero coefficients.

Max $Z=5x_1+7x_2+0s_1+0s_2$

Initial Basic feasible solution

Let $x_1=0$ and $x_2=0$ (non Basic variables) substitute in the given problem, we have $s_1=6$, $s_2=30$ (basic variables)and Max z=0.

Initial Simplex table

	C_j	5	7	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
0	$s_1=6$	-2	[3]	1	0	2 ←
0	$s_2=30$	6	1	0	1	30
z=0	Z_j	0	0	0	0	
	Z_j-C_j	-5	-7 ↑	0	0	

The diagonal element is 3.(x_2 enters the basis and s_1 leaves the basis)

Simplex iteration Rule:

(i) Divide the entire selected row by its diagonal element

(ii) All element of the selected column are zero except diagonal entry

(ii) Remaining all the position values using the formula

present position value - (corresponding selected row value) x (corresponding selected column value)/(diagonal element)

First row values for the new table

$$\frac{6}{3}=2 \quad \frac{-2}{3} \quad \frac{3}{3}=1 \quad \frac{1}{3} \quad \frac{0}{3}=0$$

Second row values for the new table

$$30-\frac{(6)(1)}{3}=28 \quad 6-\frac{(-2)(1)}{3}=-20/3 \quad 0-\frac{(1)(1)}{3}=-1/3 \quad 1-\frac{(0)(0)}{3}=1$$

First simplex table

	C_j	5	7	0	0		
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios	
7	$x_2=2$	-2/3	1	1/3	0	-	
0	$s_2=28$	[20/3]	0	-1/3	1	21/5	←
z=14	Z_j	-14/3	7	7/3	0		
	Z_j-C_j	-29/3	0	7/3	0		
		↑					

Max Z=5.0+7.2+0s_1 +0s_2=14

x_1 enters the basis and s_2 leaves the basis, the diagonal element is 20/3

First row values for the new table

$$2-\frac{(28)(-2/3)}{(20/3)}=\frac{24}{5} \quad 1-\frac{(0)(-2/3)}{20/3}=1 \quad \frac{1}{3}-\frac{(-2/3)(-1/3)}{(20/3)}=3/10 \quad 0-\frac{(-2/3)(1)}{(20/3)}=\frac{1}{10}$$

Second row values for the new table

$$\frac{28}{(20/3)}=21/5 \quad \frac{(20/3)}{(20/3)}=1 \quad \frac{0}{(21/3)}=0 \quad \frac{(-1/3)}{(20/3)}=-1/20 \quad \frac{1}{(20/3)}=3/20$$

Second simplex table

	C_j	5	7	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
7	x_2=24/5	0	1	3/10	1/10	
5	x_1=21/5	1	0	-1/20	3/20	
z=273/5	Z_j	5	7	37/20	29/20	
	Z_j-C_j	0	0	37/20	29/20	

Max Z=5x(21/5)+7x(24/5)+0s_1 +0s_2=273/5

Since $Z_j-C_j \geq 0$ for all j=1,2,3,4, therefore by the simplex procedure the above table is optimum, and the optimum solution is given by x_1=21/5, x_2=24/5 and Max z=273/5.

Step-02

In the current optimal solution all the basic variables in the basis are not integer.

Therefore the above solution is not desirable.

Step:-03

Since x_1 is the only basis variable whose value is *non-negative fraction,* so we consider the first row for generating *Gomory fractional cut.*

[Rule:- Each non integer coefficient is factorized into integer and fractional part in such a way that fractional part is positive.]

$0+24/5 = 0.\ x_1+x_2+(3/10)s_1+(1/10)s_2$

$4+4/5 \quad =(0+0).\ x_1+(1+0)x_2+(0+3/10)s_1+(0+1/10)s_2$

Therefore the Gomory's fractional cut -I is written as follows

$(4/5) \leq (3/10)\ s_1+(1/10)s_2$

$(4/5) = (3/10)\ s_1+(1/10)s_2 - s_{g1}$ (ADD Gomory surplus variable S_{g1})

$(-4/5) = (-3/10)\ s_1+(-1/10)s_2 + s_{g1}$

(Multiply by -1, so that coefficient of s_{g1} is positive)

Let us add the above equation in the last row of the optimum table and apply *dual simplex method*

First Dual simplex Table

	C_j	5	7	0	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	S_{g1}	
7	$x_2=24/5$	0	1	3/10	1/10	0	
5	$x_1=21/5$	1	0	-1/20	3/20	0	
0	$S_{g1}=-4/5$	0	0	[-3/10]	-1/10	1	←
z=273/5	Z_j	5	7	37/20	29/20	0	
	$C_j - Z_j$	0	0	-37/20	-29/20	0	
	Ratio	-	-	37/6	29/2	-	
				↑			

Ratio=$(C_j-Z_j)/y_{3j}$, calculate for all $y_{3j}<0$

Since the values of the ratio row are *positive* , therefore the above table is *not optimum*.

Always we have to select the row that contain Gomory's slack variable and column with *least positive ratio value.*

S_{g1} leaves the basis, s_1 enters the basis, diagonal element is -3/10.

First row of new table

$$24/5-\frac{(-4/5)(3/10)}{(-3/10)}=4 \quad 0-\frac{(0)(3/10)}{(-3/10)}=0 \quad 1-\frac{(0)(3/10)}{(-3/10)}=1 \quad 1/10-\frac{(3/10)(1/10)}{(-3/10)}=0$$

$$0-\frac{(3/10)(1)}{(-3/10)}=1$$

Second row of new table

$$21/5-\frac{(-4/5)(-1/20)}{(-3/10)}=13/3 \quad 1-\frac{(0)(-1/20)}{(-3/10)}=1 \quad 0-\frac{(0)(-1/20)}{(-3/10)}=0$$

$$3/20-\frac{(-1/20)(-1/10)}{(-3/10)}=1/6 \quad 0-\frac{(1)(-1/20)}{(-3/10)}=-1/6$$

Last row of the new table(divide the entire row by -3/10)

$$\frac{(-4/5)}{(-3/10)}=8/3 \quad \frac{0}{(-3/10)}=0 \quad \frac{0}{(-3/10)}=0 \quad \frac{(-3/10)}{(-3/10)}=1 \quad \frac{(-1/10)}{(-3/10)}=1/3 \quad \frac{1}{(-3/10)}=-10/3$$

Second Dual simplex Table

	C_j	5	7	0	0	0
CB_j	Basic variables	x_1	x_2	s_1	s_2	S_{g1}
7	x_2=4	0	1	0	0	1
5	x_1=13/3	1	0	0	1/6	-1/6
0	s_1=8/3	0	0	1	1/3	-10/3
z=149/3	Z_j	5	7	0	5/6	37/6
	C_j -Z_j	0	0	0	-5/6	-37/6

Since C_j -Z_j<0 for some j=1,2,3,4,5,6. Therefore the above table is *not optimum.*

Step:-03 (repetition)

Since x_1 is the basis variable whose value is *non-negative fraction,* so we consider the second row for generating *Gomory fractional cut*-II

[Rule:- Each non integer coefficient is factorized into integer and fractional part in such a way that fractional part is positive.]

$4+1/3 = 1.\ x_1+0.x_2+0.s_1+(1/6)s_2-1/6s_{g1}$

$4+1/3 \ = (1+0).\ x_1+(0+0)x_2+(0+0)s_1+(0+1/6)s_2+(-1+5/6)s_{g1}$

Therefore the Gomory's fractional cut -I is written as follows

$(1/3) \leq (1/6)\ s_2+(5/6)s_{g1}$

$(1/3) = (1/6)\ s_2+(5/6)s_{g1} - s_{g2}$ (ADD Gomory surplus variable S_{g2})

$(-1/3) = (-1/6)\ s_2+(-5/6)s_{g1} + s_{g2}$

(Multiply by -1, so that coefficient of s_{g2} is positive)

Let us add the above equation in the last row of the optimum table and *apply dual simplex method.*

Third Dual simplex Table

	C_j	5	7	0	0	0	0
CB_j	Basic variables	x_1	x_2	s_1	s_2	S_{g1}	S_{g2}
7	$x_2=4$	0	1	0	0	1	0
5	$x_1=13/3$	1	0	0	1/6	-1/6	0
0	$s_1=8/3$	0	0	1	1/3	-10/3	0
0	$S_{g2}=-1/3$	0	0	0	[-1/6]	-5/6	1
z=149/3	Z_j	5	7	0	5/6	37/6	0
	$C_j - Z_j$	0	0	0	-5/6	-37/6	0
	Ratio	-	-	-	5	37/5	-

Since few values in the ratio row are *positive*, therefore it is *not optimal,*

s_{g2} leaves the basis, and s_2 enters the basis, diagonal element is -1/6.

First row values of new table

$$4-\frac{(0)(-1/3)}{(-1/6)}=4 \quad 0-\frac{(0)(0)}{(-1/6)}=0 \quad 1-\frac{(0)(0)}{(-1/6)}=1 \quad 0-\frac{(0)(0)}{(-1/6)}=0$$

$$1-\frac{(0)(-5/6)}{(-1/6)}=1 \quad 0-\frac{(1)(0)}{(-1/6)}=0$$

Second row values of new table

$$13/3-\frac{(1/6)(-1/3)}{(-1/6)}=4 \quad 1-\frac{(1/6)(0)}{(-1/6)}=1 \quad 0-\frac{(1/6)(0)}{(-1/6)}=0 \quad 0-\frac{(1/6)(0)}{(-1/6)}=0$$

$$-1/6-\frac{(1/6)(-5/6)}{(-1/6)}=-1 \quad 0-\frac{(1)(1/6)}{(-1/6)}=1$$

Third row values of new table

$$8/3-\frac{(1/3)(-1/3)}{(-1/6)}=2 \quad 0-\frac{(1/3)(0)}{(-1/6)}=0 \quad 0-\frac{(1/3)(0)}{(-1/6)}=0 \quad 1-\frac{(1/3)(0)}{(-1/6)}=1$$

$$-10/3-\frac{(1/3)(-5/6)}{(-1/6)}=-5 \quad 0-\frac{(1)(1/3)}{(-1/6)}=2$$

Last row values of new table

$$\frac{(-1/3)}{(-1/6)}=2 \quad \frac{0}{(-1/6)}=0 \quad \frac{0}{(-1/6)}=0 \quad \frac{0}{(-1/6)}=0 \quad \frac{(-1/6)}{(-1/6)}=1 \quad \frac{(-5/6)}{(-1/6)}=5 \quad \frac{1}{(-1/6)}=-6$$

Fourth Dual Simplex Table

	C_j	5	7	0	0	0	0
CB_j	Basic variables	x_1	x_2	s_1	s_2	S_{g1}	S_{g2}
7	x_2=4	0	1	0	0	1	0
5	x_1=4	1	0	0	0	-1	1
0	s_1=2	0	0	1	0	-5	2
0	s_2=2	0	0	0	1	5	-6
z=48	Z_j	5	7	0	0	2	5
	$C_j - Z_j$	0	0	0	0	-2	-5

Since all $C_j - Z_j \le 0$ for all j, therefore the above solution *is optimal*.

Hence the optimum integer solution is x_1=4, x_2=4 and Max Z=48.

Problem:-03

Solve the following integer programming problem using Gomory's Cutting plane method, Max $Z=1.5x_1+3x_2+4x_3$, subject to conditions $2.5x_1+2x_2+4x_3\leq 12$, $2x_1+4x_2-x_3\leq 7$ and $x_1, x_2, x_3\geq 0$ and integer.

Solution:-

Let us find the optimal solution by simplex method *(ignoring integer conditions)*

Simple method

First we have to convert the inequality constraints to equality constraints by adding slack or surplus variables.

$2.5x_1+2x_2+4x_3\leq 12$,

Add slack variable $s_1\geq 0$, we have $2.5x_1+2x_2+4x_3+s_1=12$,

$2x_1+4x_2-x_3\leq 7$, Add slack variable $s_2\geq 0$, we have $2x_1+4x_2-x_3+s_2=7$,

We have to add the slack variables in the objective function with zero coefficients.

Max $Z=1.5x_1+3x_2+4x_2+0s_1+0s_2$

Initial Basic feasible solution

Let $x_1=0$, $x_2=0$ and $x_3=0$ (non Basic variables) substitute in the given problem, we have $s_1=12$, $s_2=7$ (basic variables)and Max z=0.

Initial Simplex table

	C_j	1.5	3	4	0	0		
CB_j	Basic variables	x_1	x_2	X_3	s_1	s_2	Minimum of ratios	
0	$s_1=12$	5/2	2	[4]	1	0	12/4=3	←
0	$s_2=7$	2	4	-1	0	1	-	
z=0	Z_j	0	0	0	0	0		
	Z_j-C_j	-1.5	-3	-4	0	0		
				↑				

The diagonal element is 4. (x_3 enters the basis and s_1 leaves the basis)

First row values of new table

$12/4=3, \quad (5/2)/4=5/8, \quad 4/4=1, \quad 1/4, \quad 0/4=0$

Second row values of new table

$7-\frac{(12)(-1)}{(4)}=10 \quad 2-\frac{(5/2)(-1)}{(4)}=21/8 \quad 4-\frac{(2)(-1)}{(4)}=9/2 \quad 0-\frac{(-1)(1)}{(4)}=1/4$

$1-\frac{(0)(-1)}{(4)}=1$

First Simplex table

	C_j	1.5	3	4	0	0		
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	Minimum of ratios	
4	$x_3=3$	5/8	1/2	1	1/4	0	6	
0	$s_2=10$	21/8	[9/2]	0	1/4	1	20/9	←
z=12	Z_j	5/2	2	4	1	0		
	Z_j-C_j	1	-1	0	1	0		
			↑					

second simplex table

	C_j	1.5	3	4	0	0	
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	Minimum of ratios
4	$x_3=17/9$	1/3	0	1	2/9	-1/9	
3	$x_2=20/9$	7/12	1	0	1/18	2/9	
z=128/9	Z_j	37/12	3	4	19/18	2/9	
	Z_j-C_j	19/12	0	0	19/18	2/9	

Since $Z_j-C_j \geq 0$ for all j, therefore the above table is optimum

Hence he optimum solution is $x_2 = 20/9$, $x_3=17/9$, Max z=128/9.

$_1 + 8/9 = (0+1/3)\,x_1 + (0+0)x_2 + (1+0)x_3 + (0+2/9)s_1 + (-1+8/9)s_2$

Therefore the Gomory's fractional cut -I is written as follows

$(8/9) \leq (1/3)x_1 + (2/9)\,s_1 + (8/9)s_2$

$(8/9) = (1/3)x_1 + (2/9)\,s_1 + (8/9)s_2 - S_{g1}$

(ADD Gomory surplus variable S_{g1})

$-(8/9) = -(1/3)x_1 - (2/9)\,s_1 - (8/9)s_2 + S_{g1}$

(Multiply by -1, so that coefficient of s_{g1} is positive)

First dual simple table

	C_j	1.5	3	4	0	0	0
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	S_{g1}
4	x_3=17/9	1/3	0	1	2/9	-1/9	0
3	x_2=20/9	7/12	1	0	1/18	2/9	0
0	S_{g1}=-8/9	-1/3	0	0	-2/9	[-8/9]	1
z=128/9	Z_j	37/12	3	4	19/18	2/9	0
	C_j-Z_j	-19/12	0	0	-19/18	-2/9	0
	Ratio	19/4	-	-	19/4	1/4	0

s_{g1} leaves the basis, s_2 enters the basis

Second dual simplex table

CB_j	C_j	1.5	3	4	0	0	0
	Basic variables	x_1	x_2	x_3	s_1	s_2	S_{g1}
4	$x_3=2$	3/8	0	1	1/4	0	1/8
3	$x_2=2$	1/2	1	0	0	0	1/4
0	$S_2=1$	3/8	0	0	1/4	1	-9/8
z=14	Z_j	3	3	4	1	0	5/4
	C_j-Z_j	-3/2	0	0	-1	0	-5/4

Since $Z_j-C_j \geq 0$ for all j, therefore the above table is optimum

Hence he optimum solution is $x_2= 2$, $x_3=2$, Max z=14.

Mixed Integer Programming Algorithm

Step:-01

Solve the given LP problem using simplex method, *by ignoring integer requirements of the variables.*

Step:-02

If all integer restricted basic variables have integer values, then the current solution is optimal. otherwise go to step 03.

Step:-03

Choose a row r corresponding to a basic variable x_r that has the *highest fractional value f_r* and generate a cutting plane as below

$$s_g = -f_r + \sum a_{rj} x_j + \left(\frac{f_r}{f_r - 1}\right) \sum a_{rj} x_j$$ where $0<f_r<1$.

Add this cutting plane *at the bottom* of the optimum simplex table

Step:-04

Find the new optimal solution *using dual simplex method*, and return to step 02. The process is repeated until all the restricted variables are integer.

Problem:-01

Solve the following mixed integer programming problem

Max $Z= x_1+x_2$

Subject to the constraints

$3x_1+2x_2 \leq 5$,

$x_2 \leq 2$

and $x_1, x_2 \geq 0$; x_1 is non negative integer.

Solution:-

Step:-01

Let us find the optimal solution by simplex method *(ignoring integer conditions)*

Simple method

First we have to convert the inequality constraints to equality constraints by adding slack or surplus variables.

$3x_1+2x_2 \leq 5$, Add slack variable $s_1 \geq 0$, we have $3x_1+2x_2+s_1=5$,

$x_2 \leq 2$, Add slack variable $s_2 \geq 0$, we have $x_2+s_2=2$,

Max $Z=x_1+x_2+0s_1+0s_2$

Initial Basic feasible solution

Let $x_1=0$ and $x_2=0$ (non Basic variables) substitute in the given problem, we have *$s_1=5$, $s_2=2$ (basic variables)and max $z=0$.*

Initial Simple table

	C_j	1	1	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
0	$s_1=5$	[3]	2	1	0	5/3 ←
0	$s_2=2$	0	1	0	1	-
z=0	Z_j	0	0	0	0	
	Z_j-C_j	-1 ↑	-1	0		

Since some Z_j-C_j are <0, therefore the above solution is *not optimal*.

Diagonal element is 3. x_1 enters the basis and s_1 leaves the basis

First row values

5/3, 3/3=1 2/3 1/3 0/3=0

Second row values

2-(5)(0)/3=2   1-(0)(2)/3=1   0-(0)(1)/3=0   1-(0)(0)/3=1

First simplex table

	C_j	1	1	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
1	x_1=5/3	1	2/3	1/3	0	5/2
0	s_2=2	0	[1]	0	1	2
z=5/3	Z_j	1	2/3	1/3	0	
	Z_j-C_j	0	-1/3	1/3	0	

Since some Z_j-C_j are <0, therefore the above solution is *not optimal*.

Diagonal element is 1, x_2 enters the basis and s_2 leaves the basis

First row values

5/3-(2)(2/3)/1=1/3,   1-(0)(2/3)/1=1   1/3-(2/3)(0)=1/3

0-(1)(2/3)=-2/3

Second row values

2/1=2   0/1=0   1/1=1   0/1=1   1/1=1

Second simplex table

	C_j	1	1	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	Minimum of ratios
1	x_1=1/3	1	0	1/3	-2/3	
1	x_2=2	0	1	0	1	
z=7/3	Z_j	1	1	1/3	1/3	
	Z_j-C_j	0	0	1/3	1/3	

Since all Z_j-C_j values are non-negative, therefore the above solution is *optimal.*

The optimal solution is $x_1=1/3$, $x_2=2$ *and Max* $Z=7/2$

Step-02

In the current optimal solution the basic variable x_1 *not integer*.

Therefore the above solution is *not desirable*.

Step:-03

Since x_1 is the only basis variable whose value is *non-negative fraction,* so we consider the first row for generating *Gomory fractional cut.*

[Rule:- Each non integer coefficient is factorized into integer and fractional part in such a way that fractional part is positive.]

$1/3=x_1+0x_2+1/3s_1-2/3s_2$

$(0+1/3)=(1+0)x_1+(0+1/3)s_1+(-1+1/3)s_2$

$1/3\leq 0+(1/3)s_1+(1/3)s_2$ (Gomory fractional cut-I)

$1/3=(1/3)s_1+(1/3)s_2-s_{g1}$ (Add Gomory surplus variable s_{g1})

$-1/3=-(1/3)s_1-(1/3)s_2+s_{g1}$

(Multiply by -1, so that coefficient of s_{g1} is positive)

Add this in the above table as a last row, we have

First Dual Simplex Table

	C_j	1	1	0	0	0	
CB_j	Basic variables	x_1	x_2	s_1	s_2	s_{g1}	
1	$x_1=1/3$	1	0	1/3	-2/3	0	
1	$x_2=2$	0	1	0	1	0	
0	$s_{g1}=-1/3$	0	0	[-1/3]	-1/3	1	←
z=7/3	Z_j	1	1	1/3	1/3	0	
	C_j- Z_j	0	0	-1/3	-1/3	0	
	Ratio	-	-	1	1	-	
				↑			

Ratio=$(C_j - Z_j)/y_{3j}$, where $y_{3j}<0$.

s_{g1} leave the basis and s_1 enters the basis, diagonal element is -1/3

Second Dual Simplex Table

	C_j	1	1	0	0	0
CB_j	Basic variables	x_1	x_2	s_1	s_2	s_{g1}
1	$x_1=0$	1	0	0	-1	1
1	$x_2=2$	0	1	0	1	0
0	$s_1=1$	0	0	1	1	-3
z=2	Z_j	1	1	0	0	1
	$C_j- Z_j$	0	0	0	0	-1

Since all $C_j- Z_j \leq 0$, therefore the above solution is optimal.

Hence the mixed integer optimum solution is *$x_1=0$, $x_2=2$ and Max $Z=2$.*

Problem:02

Solve the following mixed integer problem

Max $Z=4x_1+6x_2+2x_3$

Subject to constraints

$4x_1-4x_2 \leq 5,$

$-x_1+6x_2 \leq 5,$

$-x_1+x_2+x_3 \leq 5$

and $x_1, x_2, x_3 \geq 0$; x_1, x_3 are integers.

Solution:-

Simplex method

$4x_1-4x_2 \leq 5$, Add slack variable s_1, we have $4x_1-4x_2+s_1=5$,

$-x_1+6x_2 \leq 5$, Add slack variable s_2, we have $-x_1+6x_2+s_2=5$,

$-x_1+x_2+x_3 \leq 5$, Add slack variable s_3, we have $-x_1+x_2+x_3+s_3=5$.

Max $Z=4x_1+6x_2+2x_3+0s_1+0s_2+0s_3$

IBFS $x_1=0, x_2=0, x_3=0, s_1=5, s_2=5, s_3=5$, Max $z=0$.

Initial Simplex table

	C_j	4	6	2	0	0	0	
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
0	$s_1=5$	4	-4	0	1	0	0	-
0	$s_2=5$	-1	[6]	0	0	1	0	5/6
0	$s_3=5$	-1	1	1	0	0	1	5
z=0	Z_j	0	0	0	0	0	0	
	$Z_j - C_j$	-4	-6	-2	0	0	0	

Since some $Z_j - C_j$ values are negative, therefore the above is *not a optimal table.*

i.e x_2 enters the basis and s_2 leaves the basis, diagonal element is 6

First row values

5-(5)(-4)/6=25/3, 4-(-1)(-4)/6=10/3 0-(-4)(0)/6=0

1-(0)(-4)/6=1 0-(1)(-4)/6=2/3 0-(0)(-4)/6=0

Second row values

5/6 -1/6 6/6=1 0/6=0 0/6=0 1/6 0/6

Third row values

5-(5)(1)/6=25/6 -1-(-1)(1)/6=-5/6 1-(0)(1)/6=1

0-(0)(1)/6=0 0-(1)(1)/6=-1/6 1-(0)(1)/6=1

First simplex table

	C_j	4	6	2	0	0	0	
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
0	s_1=25/3	[10/3]	0	0	1	2/3	0	25/10 ←
6	x_2=5/6	-1/6	1	0	0	1/6	0	
0	s_3=25/6	-5/6	0	1	0	-1/6	1	
z=5	Z_j	-1	6	0	0	1	0	
	Z_j -C_j	-5 ↑	0	-2	0	1	0	

Since some Z_j -C_j values are negative, therefore the above table *is not optimal.*

x_1 enters the basis, s_1 leaves the basis, diagonal element is 10/3.

First row values

(25/3)/(10/3)=5/2, (10/3)/(10/3)=1 0/(10/3)=0

0/(10/3)=0 1/(10/3)=3/10 (2/3)/(10/3)=1/5

0-(25/3)(0)/(10/3)=0

Second row values

(5/6)-(25/3)(-1/6)/(10/3)=5/4 1-(-1/6)(0)/=1

0-(0)(-1/6)/(10/3)=0 0-(1)(-1/6)/(10/3)=1/20

(1/6)-(2/3)(-1/6)/(10/3)=1/5 0-(0)(-1/6)/(10/3)=0

Third row values

(25/6)-(25/3)(-5/6)/(10/3)=25/4 0-(-5/6)(0)/(10/3)=0

1-(0)(-5/6)/(10/3)=1 0-(-5/6)(1)/(10/3)=1/4

-1/6-(-5/6)(2/3)/(10/3)=0 1-(0)(-5/6)/(10/3)=1

Second simplex table

	C_j	4	6	2	0	0	0	
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
4	x_1=5/2	1	0	0	3/10	1/5	0	-
6	x_2=5/4	0	1	0	1/20	1/5	0	-
0	s_3=25/4	0	0	[1]	1/4	0	1	25/12 ←
z=35/2	Z_j	4	6	0	3/2	2	0	
	Z_j -C_j	0	0	-2 ↑	3/2	2	0	

Since some Z_j -C_j values are negative, therefore the above table *is not optimal.*

x_3 enters the basis, s_3 leaves the basis, diagonal element is 1.

First row values

(5/2)-(0)(25/12)/1=5/2 1-(0)(0)/1=1 0-(0)(0)/1=0

3/10-(0)(1/4)/1=3/10 1/5-(0)(0)/1=1/5 0-(1)(0)/1=0

Second row values

(5/4) -(0)(25/12)/1=5/4 0-(0)(0)/1=0 1-(0)(0)/1=1

(1/20)-(0)(0)/1=1/20 1/5-(0)(0)/1=1/5 0-(0)(1)/1=0

Third row values

(25/4)/1=25/4 0/1=0 0/1=0 1/1=1

(1/4)/1=1/4 0/1=0 1/1=1

Third simplex table

	C_j	4	6	2	0	0	0	
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Ratio
4	x_1=5/2	1	0	0	3/10	1/5	0	-
6	x_2=5/4	0	1	0	1/20	1/5	0	-
2	x_3=25/4	0	0	1	1/4	0	1	-
z=30	Z_j	4	6	2	2	2	2	
	Z_j -C_j	0	0	0	2	2	0	

Since some Z_j -C_j values are either zero or positive, therefore the above table is *optimal*.

In the above optimal solution x_1 and x_3 are not integer, therefore the current optimal solution is *not desirable* .

Let us consider the fractional cut (5/2=2+1/2) (25/4=6+1/4)

$5/2=x_1+(3/10)s_1+(1/5)s_2$

$(2+1/2)=(1+0)x_1+(0+3/10)s_1+(0+1/5)s_2$

$(1/2)\leq(3/10)s_1+(1/5)s_2$ (Fractional cut I)

$(1/2)=(3/10)s_1+(1/5)s_2-s_{g1}$ (Add Gomory slack variable s_{g1})

$-1/2=-3/10s_1-1/5s_2+s_{g1}$ (Multiply by -1)

Add the above equation in the optimum simplex table and solve using dual simplex methods, we have

First Dual Simplex table

CB_j	C_j	4	6	2	0	0	0	
	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	s_{g1}
4	$x_1=5/2$	1	0	0	3/10	1/5	0	0
6	$x_2=5/4$	0	1	0	1/20	1/5	0	0
2	$x_3=25/4$	0	0	1	1/4	0	1	0
0	$s_{g1}=-1/2$	0	0	0	[-3/10]	-1/5	0	1
z=30	Z_j	4	6	2	2	2	2	0
	$C_j - Z_j$	0	0	0	-2	-2	-2	0
	Ratio	-	-	-	20/3	10	-	-

Since some values of ratio are positive, therefore the above *it not optimal.*

s_1 enters the basis, s_{g1} leaves the basis, diagonal element -3/10.

First row values

(5/2)-(3/10)(-1/2)/(-3/10)=2 1-(3/10)(0)/(-3/10)=1

0-(3/10)(0)/(-3/10)=0 0-(3/10)(0)/(-3/10)=0

(1/5)-(3/10)(-1/5)/(-3/10)=0 0-(0)(3/10)/(-3/10)=0

0-(1)(3/10)/(-3/10)=1

Second row values

(5/4)-(-1/2)(1/20)/(-3/10)=7/6 0-(0)(1/20)/(-3/10)=0

1-(1/20)(0)/(-3/10)=1 0-(1/20)(0)/(-3/10)=0

(1/5)-(1/20)(-1/5)/(-3/10)=1/6 0-(0)(1/20)/(-3/10)=0

0-(1/20)(1)/(-3/10)=1/6

Third row values

(25/12)-(-1/2)(1/4)/(-3/10)=5/3 0-(0)(1/4)/(-3/10)=0

0-(0)(1/4)/(-3/10)=0 1-(0)(1/4)/(-3/10)=1

0-(-1/5)(1/4)/(-3/10)=-1/6 1-(0)(1/4)/(-3/10)=1

0-(1)(1/4)/(-3/10)=5/6

Fourth row values

(-1/2)/(-3/10)=5/3 0/(-3/10)=0 0/(-3/10)=0 0/(-3/10)=0

(-3/10)/(-3/10)=1 (-1/5)/(-3/10)=2/3 0/(-3/10)=0

1/(-3/10)=-10/3

Second dual simplex Table

	C_j	4	6	2	0	0	0	0
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	s_{g1}
4	$x_1=2$	1	0	0	0	0	0	1
6	$x_2=7/6$	0	1	0	0	1/6	0	1/6
2	$x_3=35/6$	0	0	1	0	-1/6	1	5/6
0	$s_1=5/3$	0	0	0	1	2/3	0	-10/3
z=80/3	Z_j	4	6	2	0	2/3	2	20/3
	$C_j - Z_j$	0	0	0	0	-2/3	-2	-20/3

Since all $C_j - Z_j \leq 0$, therefore the above table is *optimal*, but the basic variable *x_3 is not integer*.

Let us add fractional cut in the above table

$35/6 = x_3 - (1/6)s_2 + s_3 + (5/6)s_{g1}$

$(5+5/6) = (1+0)x_3 + (-1+5/6)s_2 + (1+0)s_3 + (0+5/6)s_{g1}$

$(5/6) \leq (5/6)s_2 + (5/6)s_{g1}$ (Fractional cut-II)

$(5/6) = (5/6)s_2 + (5/6)s_{g1} - s_{g2}$ (Add Gomory slack variable s_{g2})

$(-5/6) = -(5/6)s_2 - (5/6)s_{g1} + s_{g2}$ (Multiply by -1)

Add the this equation at the end of the above optimum table.

Third Dual Simplex Table

	C_j	4	6	2	0	0	0		
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	s_{g1}	s_{g2}
4	$x_1=2$	1	0	0	0	0	0	1	0
6	$x_2=7/6$	0	1	0	0	1/6	0	1/6	0
2	$x_3=35/6$	0	0	1	0	-1/6	1	5/6	0
0	$s_1=5/3$	0	0	0	1	2/3	0	-10/3	0
0	$s_{g2}=-5/6$	0	0	0	0	[-5/6]	0	-5/6	1
z=80/3	Z_j	4	6	2	0	2/3	2	20/3	0
	$C_j - Z_j$	0	0	0	0	-2/3	-2	-20/3	0
	Ratio	-	-	-	-	4/5	-	8	-

Some values in the ratio row are positive, therefore the above table is *not optimal.*

s_{g2} leaves the basis, s_2 enters the basis, diagonal element -5/6.

First row values

2-(0)(-5/6)/(-5/6)=2   1-(0)(0)/(-5/6)=1   0-(0)(0)/(-5/6)=0

0-(0)(0)/(-5/6)=0   0-(0)(0)/(-5/6)=0   0-(0)(0)/(-5/6)=0

1-(0)(-5/6)/(-5/6)=1   0-(0)(1)/(-5/6)=0

Second row values

(7/6)-(1/6)(-5/6)/(-5/6)=1   0-(0)(1/6)/(-5/6)=0

1-(1/6)(0)/(-5/6)=1   0-(1/6)(0)/(-5/6)=0

0-(0)(1/6)/(-5/6)=0   0-(1/6)(0)/(-5/6)=0

1/6-(1/6)(-5/6)/(-5/6)=0   0-(1/6)(1)/(-5/6)=1/5

Third row values

(5/3)-(-1/6)(-5/6)/(-5/6)=11/6   0-(-1/6)(0)/(-5/6)=0

0-(-1/6)(0)/(-5/6)=0   1-(-1/6)(0)/(-5/6)=1

0-(-1/6)(0)/(-5/6)=0   1-(-1/6)(0)/(-5/6)=1

(5/6)-(-1/6)(-5/6)/(-5/6)=1   0-(-1/6)(1)/(-5/6)=-1/5

Fourth row values

(5/3)-(2/3)(-5/6)/(-5/6)=1   0-(2/3)(0)/(-5/6)=0

0-(2/3)(0)/(-5/6)=0   0-(2/3)(0)/(-5/6)=0

1-(2/3)(0)/(-5/6)=1   0-(2/3)(0)/(-5/6)=0

(-10/3)-(2/3)(-5/6)/(-5/6)=-4   0-(2/3)(1)/(-5/6)=4/5

Fifth row values

(-5/6)/(-5/6)=1   0/(-5/6)=0,   0/(-5/6)=0,   0/(-5/6)=0

0/(-5/6)=0,   (-5/6)/(-5/6)=1   0/(-5/6)=0

(-5/6)/(-5/6)=1   1/(-5/6)=-6/5

Fourth Dual Simplex Table

	C_j	4	6	2	0	0	0		
CB_j	Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	s_{g1}	s_{g2}
4	$x_1=2$	1	0	0	0	0	0	1	0
6	$x_2=1$	0	1	0	0	0	0	0	1/5
2	$x_3=6$	0	0	1	0	0	1	1	-1/5
0	$s_1=1$	0	0	0	1	0	0	-4	4/5
0	$s_2=1$	0	0	0	0	1	0	1	-6/5
z=26	Z_j	4	6	2	0	0	2	6	4/5
	$C_j - Z_j$	0	0	0	0	0	-2	-6	-4/5

Since all $C_j - Z_j \leq 0$, therefore the above table *is optimal*.

Hence the mixed integer optimal solution is given by $x_1=2$, $x_2=1$, $x_3=6$ *and Max* $Z=4(2)+6(1)+2(6)=26$.

Branch and Bound Method

- It was first developed by *A H Land and AG Doig.*
- It can be used to solve all integer, mixed integer and zero-one integer programming problems
- It partition the feasible solution region in to *smaller regions* until the optimal solution obtained.
- It starts by imposing *bounds on* the value of the objective function that helps the sub problem to be eliminated from the consideration when the optimum solution has been found.

Procedure of Branch and Bound Algorithm

Step:-01(Initialization)

Consider the following all integer programming problem

Max $Z=c_1x_1+c_2x_2+.....c_nx_n$

Subject to the constraints

$a_{11}x_1+a_{12}x_2+......+a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+......+a_{2n}x_n=b_{2..}$

.

$a_{m1}x_1+a_{m2}x_2+......+a_{mn}x_n=b_{1m}$

and $x_j \geq 0$ and non negative integer.

Obtain the optimal solution of the given problem ignoring integer restriction on the variables

(i) If the solution to this LP problem (say LP-A) is infeasible or unbounded, the solution to the all integer programming problem is also infeasible or unbounded , as the case may be.

(ii) If the solution satisfies the integer restriction, the optimal integer solution has been obtained. If one or more basic variables do not satisfies integer requirement, then go to step 2. Let the optimal value of the objective function of LP-A be $Z_{1.}$ This value provides an initial upper bound on objective function value and it is denoted by $Z_{U.}$

(iii) Find the feasible value by rounding off each variable value. The value of the objective function so obtained is the least upper bound and it is denoted by $Z_{L.}$

Step:-02(Branching step)

(i) Let x_k be one basic variable which does not have an integer value and also has the largest fractional value

(ii) Branch (or partition) the LP-A into two new LP sub problems (also called nodes) based on integer values of x_k that are immediately above and below its non integer value. That is, it is portioned by adding two mutually exclusive constraints.:

$$x_k \le [x_k] \text{ and } x_k \ge [x_k]+1$$

to the original LP problem, Here $[x_k]$ is the integer portion of the current non-integer value of the variable x_k. This is obviously is done to exclude the non integer value of the variable x_k. The two new Lp sub problems are as follows:

<u>LP Sub problem B</u>

Max Z= $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$

$x_k \le [x_k]$

and $x_j \ge 0$

<u>LP Sub Problem C</u>

Max Z= $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i$

$x_k \ge [x_k]+1$

and $x_j \ge 0$

Step:-03(Bound step)

Obtain the optimal solution of sub problems B and C. Let the optimal value of the objective function of LP-B be Z_2 and that of LP-C be Z_3. The best integer solution value becomes the lower bound on the integer LP problem objective function value (initially this is the rounded off value). Le the lower bound be denoted by Z_L.

Step:-03(Fathoming step) Examining the solution of both LP-B and LP-C

(i) If the sub problem yields an infeasible solution , then terminate the branch.

(ii) If the sub problems yields the feasible solution, but not integer solution then go to step 2.

(iii) If the sub problem yields the integer feasible solution, then examine the value of the objective function. If the value is equal to upper bound, then the optimal solution has been obtained. But if it is not equal to upper bound but exceed the lower bound, then this value is considered as new upper bound and return to step 2. Finally, If it is less than the lower bound terminate the branch.

Step 05 (Termination)

procedure of branching and bounded continues, until no sub problem remains to be examined. At this stage, the integer solution corresponding to the current lower bound is optimal all integer programming problem.

Problem:-01

Solve the following all integer programming problem using branch and bound method,

Max $Z=2x_1+3x_2$,

Subject to the conditions

$6x_1+5x_2 \leq 25$,

$x_1+3x_2 \leq 10$

and $x_1, x_2 \geq 0$ and integers

Solution:-

Step:-01

Relaxing the integer condition, Let us find the optimal solution to the given LP problem by graphical method

Graphical Method

Let us draw the line $6x_1+4x_2=25$-----(1)

Let $x_1=0$, $6(0)+5x_2=25$ $\Rightarrow x_2=25/4$ i.e (0,5)

Let $x_2=0$, $6x_1+5(0)=25$ $\Rightarrow x_1=25/6$ i.e(4.16,0)

Let us draw the line $x_1+3x_2=10$------(2)

Let $x_1=0$, $(0)+3x_2=10$ $\Rightarrow x_2=10/3$ i.e (0,3.33)

Let $x_2=0$, $x_1+3(0)=10$ $\Rightarrow x_1=10$ i.e(10,0)

To find the intersection point

(1)x1=> $6x_1+5x_2=25$

(2)x6=> $6x_1+18x_2=60$

(Subtract) ------------------

$-13x_2=-35$ $\quad\Rightarrow x_2=35/13$

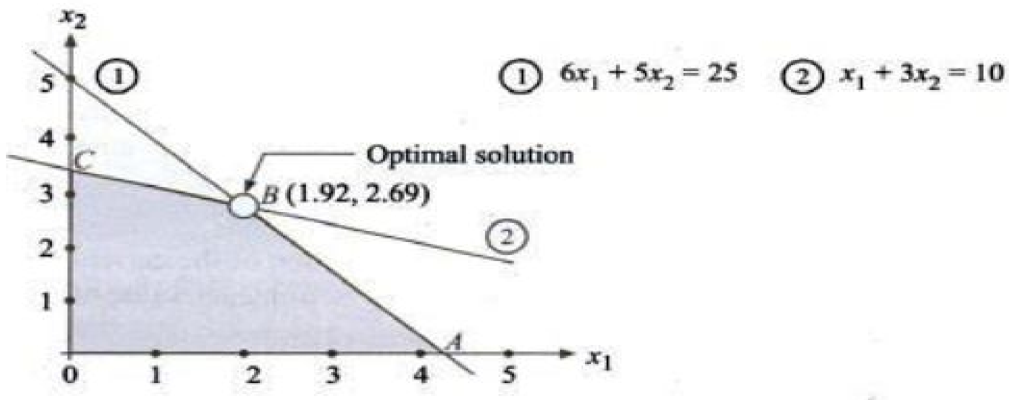
Sub x_2 values in equation (2), we have $x_1=25/13$

i.e the intersection point is (1.92,2.69)

At (4.16,0) Max $Z=2(4.16)+3(0)=8.32$

At (0,3.33) Max $Z=2(0)+3(3.33)=9.99$

At (1.92,2.69) Max $Z=2(1.92)+3(2.69)=11.91$



The optimum solution is *$x_1=1.92$, $x_2=2.69$ and Max $Z_1=11.91$.*

Let $Z_U=11.91$ be the *initial upper* bound.

The feasible solution by rounding off the solution gives the *initial lower bound* $Z_L=11$ (rounding off $x_1=1$, $x_2=3$). The optimal solution lies between these two bounds.

Step:-02(Branching step)

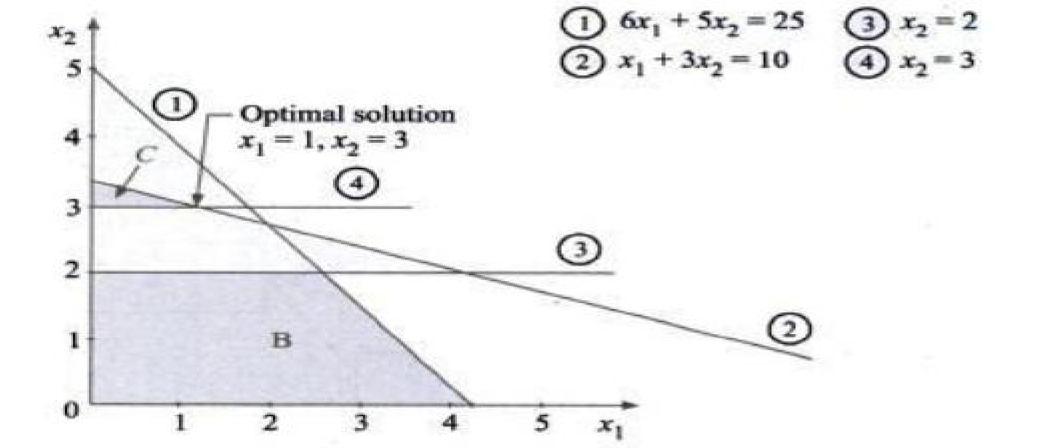
Here both x_1 and x_2 are not integers, therefore we can select variable for branching with highest fractional value, here x_2 has more fractional value compared to x_1.

Solving the variable x_2 for branching, divide the given problem into two *sub problem A and B* to eliminate the fractional part of $x_2=2.69$,

The new constraints to be added are $x_2 \leq 2$ *and* $x_2 \geq 3$.

LP sub problem B	LP sub problem C
Max $Z=2x_1+3x_2$,	Max $Z=2x_1+3x_2$,
Subject to the conditions	Subject to the conditions
$6x_1+5x_2 \leq 25$,	$6x_1+5x_2 \leq 25$,
$x_1+3x_2 \leq 10$	$x_1+3x_2 \leq 10$
$x_2 \leq 2$	$x_2 \geq 3$
and $x_1, x_2 \geq 0$ and integers	and $x_1, x_2 \geq 0$ and integers
Draw the line $x_2=2$-------(3)	*Draw the line $x_2=3$--------(4)*
It is the line parallel to x_1 axis and passes through (0,2)	It is a line parallel to x_1 axis and passes through (0,3)

The corresponding graph is given by



The solution to the sub problem B is given by equation (1) and (3) Sub $x_2=2$ in equation (1), we have $6x_1+5(2)=25$ $\Rightarrow x_1=5/2$ i.e (2.5, 2) and *Max $Z_2=2(2.5)+3(2)=11$*	The solution to the sub problem C is given by equation (2) and (4) Sub $x_2=3$ in equation (2), we have $x_1+3(3)=10$ $\Rightarrow x_1=1$ i.e (1, 3) and *Max $Z_3=2(1)+3(3)=11$*

Step:-03 (Bound step)

Rule :-The best integer solution gives the lower bound.

Here the best integer solution is $x_1=1$, $x_2=3$ and Max $Z_3=11$

The value $Z_L=11$ is the *lower bound* for the given LP problem.

(i.e the value of the objective function in the sub sequence steps cannot be less than this)

Step:-04(Fathoming step)

Both the sub problems B and C gives same Z value.

In the sub problem C both the variables x_1 and x_2 are integer, so there is *no need to branch the sub problem C further*.

Since $Z_L \leq Z_3 < Z_U$ $(11 \leq 11 < 11.91)$, therefore the new upper bound is $Z_U=11$.

In the sub problem B x_1 is not an integer and since $Z_L \leq Z_2$ $(11 \leq 11)$.

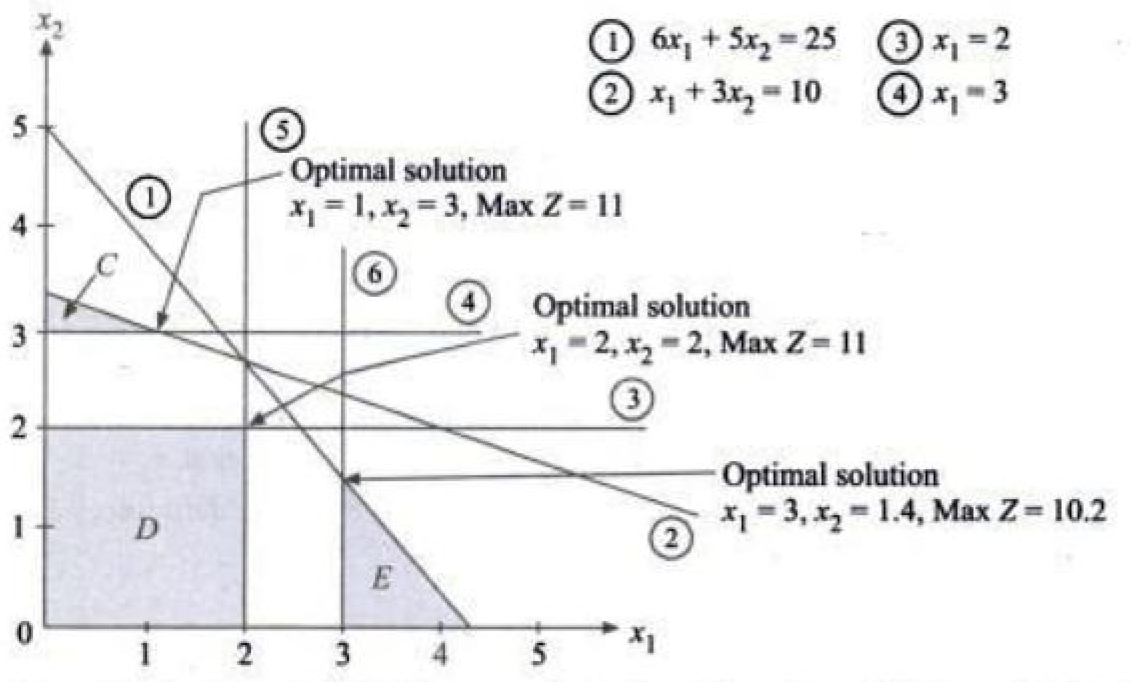
Therefore the sub problem B can be branch further.

Step:-05 (Branch step)

Let us divide the sub problem B into two sub problem namely *D and E by taking variable $x_1=2.5$.*

The new constraints to be added are $x_1 \leq 2$ *and* $x_1 \geq 3$.

LP sub problem D	LP sub problem E
Max $Z=2x_1+3x_2$,	Max $Z=2x_1+3x_2$,
Subject to the conditions	Subject to the conditions
$6x_1+5x_2 \leq 25$,	$6x_1+5x_2 \leq 25$,
$x_1+3x_2 \leq 10$	$x_1+3x_2 \leq 10$
$x_2 \leq 2$	$x_2 \leq 2$
$x_1 \leq 2$	$x_1 \geq 3$
and $x_1, x_2 \geq 0$ and integers	and $x_1, x_2 \geq 0$ and integers
Draw the line $x_1=2$-------(5)	*Draw the line $x_1=3$------(6)*
It is the line parallel to x_2 axis and passes through (2,0)	It is the line parallel to x_2 axis and passes through (3,0)



The solution to the sub problem D is given by equation (3) and (5) i.e $x_1=2$ and $x_2=2$ *Max $Z_4=2(2)+3(2)=10$*	The solution to the sub problem E is given by equation (1) and (6) Sub $x_1=3$ in equation (1), we have $6(3)+5x_2=25$ $\Rightarrow x_2=1.4$ i.e (3, 1.4) and *Max $Z_5=2(3)+3(1.4)=10.2$*

Step:-05(Bound step)

Rule :-The best integer solution gives the lower bound.

Here the best integer solution is $x_1=2$, $x_2=2$ and Max $Z_4=10$

Step:-06(Bound step)

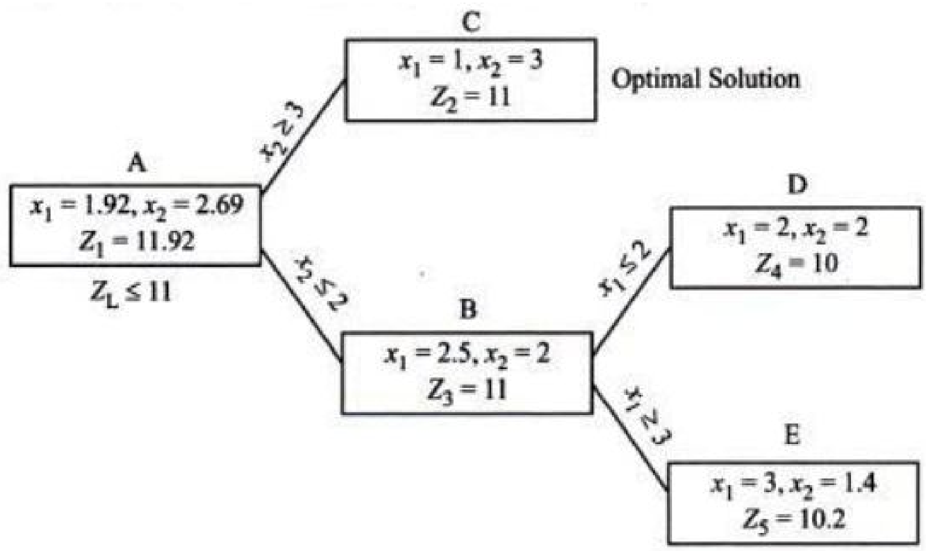
The solution to the LP sub problem D is integer feasible, but $Z_4< Z_L$ (10<11), and since value of lower bound remain unchanged ($Z_L=11$). therefore the sub problem D is not considered for further branching.

Since solution of sub problem E is non integer (x_2 is not integer), So it can be further branched with variable x_2, but $Z_5 < Z_L$ (10.2<11), therefore sub problem E is not considered for further branching.

Thus the integer solution corresponding to the current lower bound is the optimal solution.

Hence the integer optimal solution is corresponding to sub problem C and is given by $x_1=1$, $x_2=3$ and Max Z=11

The entire branch and bound procedure can be represented by following enumeration tree. Each node is a sub problem.



Note:-

(i) Suppose if both the sub problem gives non integer solution, then always the sub problem which has maximum objective value has to branched further.

(ii) Suppose if one of the sub problem has no feasible solution and other has feasible solution but not integer and any other sub problem of same kind exist, always select the sub problem which has maximum objective function value for further branching.

Dynamic programming problem

Introduction:-

Dynamic programming is a mathematical technique dealing with the optimization of *multistage decision processes*. It provides a systematic procedure for determining the optimal combination decision.

In contrast to linear programming, there does not exist a standard mathematical formulation of dynamic programming problem. The word programming has been used in the mathematical sense of selecting (planning) an *optimal allocation of resources*, and it is *'dynamic'* as it particularly useful for problems where decisions are taken at *several stages*, such as everyday or every week.

Dynamic Programming Technique

- Dynamic programming is a technique that allow us to break up difficult problems into sequence of sub problems

- It start with a small sub problem and finds the optimal solution for this smaller problem.
- It then gradually enlarges the problem, finding the current optimal solution from the preceding one, until the original problem is solved in its entirely.

Richard Bellman developed this technique in early 1950 and invented its name. It is also known as *recursive optimization*.

Backward Recursive Equation(function)

Let $f_i(x_j)$ be the shortest distance to node j at stage i.

Let $d(x_{j-1}, x_j)$ be the distance between x_{j-1} and x_j.

The recursive equation is given by

$$f_i(x_j) = \min_{\substack{\text{all feasible} \\ (x_{j-1}, x_j) \text{ routes}}} \{d(x_{j-1}, x_j) + f_{i-1}(x_{j-1})\}$$ where i=1,2,3...n (stages) and j=1,2,3...m (states)

Note:-

In backward recursive equation we calculate $f_i(x_j)$ is obtained by substituting the value of $f_{i-1}(x_{j-1})$ (previous stage value)

Forward Recursive Equation(function)

Let $f_i(x_j)$ be the shortest distance to node j at stage i.

Let $d(x_{j-1}, x_j)$ be the distance between x_{j-1} and x_j.

The forward recursive equation is given by

$$f_i(x_j) = \min_{\substack{\text{all feasible} \\ (x_{j-1}, x_j) \text{ routes}}} \{d(x_j, x_{j+1}) + f_{i+1}(x_{j+1})\}$$ where i=1,2,3...n (stages) and j=1,2,3...m (states)

Note:-

In forward recursive equation we calculate $f_i(x_j)$ by substituting the value of $f_{i+1}(x_{j+1})$ (next stage value)

Bellman's Principle of Optimality

An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision process.

In greedy method only one decision sequence is generated.

In dynamic programming many decision sequences may be generated

Forward Recursion

Forward recursion in which computation proceeds starts from the stage 1 and ending at stage n.

Backward Recursion

Backward recursion in which computation proceeds starts from the stage n and ending at stage 1.

In dynamic programming problem, the solution process is initiated by determining the optimal policy for the last stage, which we label as stage 1, since we analyze it first, with the state transformation function and recursive criterion function developed for the particular problem. We then work backward stage by stage to find the optimal policy for each state of every stage i, given the optimal policy for each state of stage i-1. The optimal policy for the entire problem is determined when this process terminates at the initial stage. This procedure is called backward recursion.

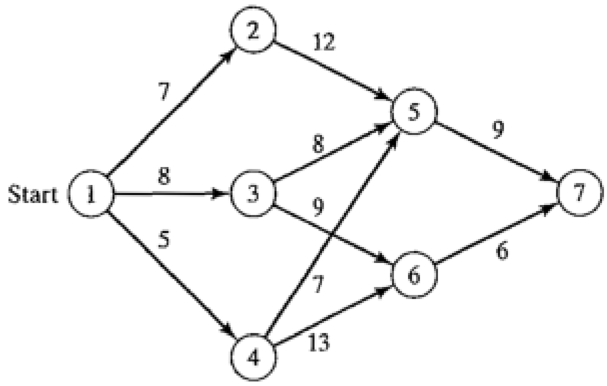
Note :-

- Both forward and backward procedure yields the same solution.
- Forward procedure seems to be more logic

➢ DP invariably uses backward procedure, the reason is backward procedure more efficient computationally.

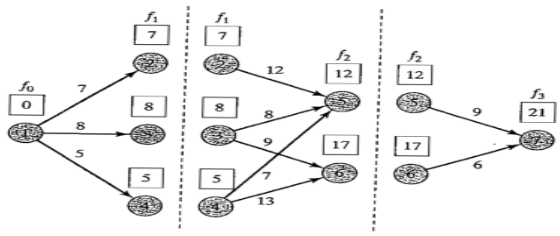
SHORTEST ROUTE PROBLEM

Suppose that you want to select the shortest highway route between two cities. The network in the following figure provides the possible route between the starting city at node 1, and the ending city at node 7.



We can solve this problem by exhaustively enumerating all such routes between node 1 to node 7 (Here there are five such routes namely 1-2-5-7, 1-3-6-7, 1-3-5-7, 1-4-5-7, 1-4-6-7). However in large network exhaustive enumeration is intractable computationally.

We first decompose this problem into stages as delineated by the vertical dashed lines in following figure



(i) Solve using forward recursion procedure

Optimum solution at Stage `1'

In stage one ending nodes are 2,3,4, this nodes can be reached from starting node 1 only.

Shortest distance from node 1 to node 2 is 7

shortest distance from node 1 to node 3 is 8

shortest distance from node 1 to node 4 is 5

Optimum solution at Stage `2'

In stage one ending nodes are 5,6.

Here the node 5 can be reached from three nodes 2,3,4.

the node 6 can be reached from two nodes 3,4.

Shortest distance from node 1 to node 5 are 7+12=19, 8+8=16, 5+7=12 corresponding to the following routes 1-2-5, 1-3-5, 1-4-5. Here the minimum value is 12 corresponding to the routes 1-4-5

Shortest distance from node 1 to node 6 is 8+9=17, 5+13=18 corresponding to the following routes 1-3-6, 1-4-6. Here the minimum value is 17 corresponding to the routes 1-3-6.

Optimum solution at Stage `3'

In stage one ending nodes is 7.

Here the node 7 can be reached from three nodes 5,6.

Shortest distance from node 1 to node 7 are 12+9=21, 17+6=23 corresponding to the following routes 1-4-5-7, 1-3-6-7. Here the minimum value is 21 corresponding to the routes 1-4-5-7.

Hence stage 3 shows that the shortest distance between node 1 to 7 is 21 miles,

(ii) Solve using Back word recursion.

Since there are three stages, therefore i=1,2,3.

and there are 7 states, therefore j=1,2,3,4,5,6,7.

Stage

Each point in the problem where a decision must be made.

Example:-

(i) In Shortest Root Problem

The stage may be group of cities with a common property (minimum number of areas from the destination)

(ii) In Sales Man Allocation Problem

Each territory represents a stage.

State

Information describing the problem at each stage, generally in the form of *specific value* of state variables.

Example:-

(i) In Shortest Root Problem

The state at any stage was a specific city.

(ii) In Sales Man Allocation Problem

The state was the *specific value* of the number of salesmen still available, Here the number of salesman is a state variable.

Policy A

A decision making rule which, at any stage, permits a feasible sequence of decisions. In effect, a policy transforms

Example:-

(i) In Shortest Root Problem

The policy at any stage of the problem was the selection of the routes to the next group of cities.

(ii) In Sales Man Allocation Problem

The policy at any stage of the problem would be the allocation of some specific number of available salesman to the territory represented by that stage.

Optimal Policy A

A policy which optimizes the value of a criterion, objective, or return function. Stating in any given state of any stage., the optimal policy depends only upon that state and not upon how it was reached.

In other words, the optimal decision at any state is in no way dependent on the previous history of the system.

i.e The future decisions for the remaining stages will constitute an optimal policy regardless of the policy adopted in previous stage.

State Transformation Function

The state transformation function T_i transforms state S_i to state S_{i-1}, given decision D_i.

i.e $T_i(S_i,D_i)=S_{i-1}$, Moving backward to the system.

Note

There are two differences between Linear programming and Dynamic programming problem.

(i) There is no common method available for solving all dynamic programming problem, like simplex method available for linear programming problem.

❖ **(ii)** Linear programming is a method that gives single (one time period) solutions. But dynamic programming has the power to determine the optimal solution over a one year time horizon by breaking the problem into twelve smaller one-month time horizon problem and solve each of the optimally. ie. in DP many decision making sequence may be generated.

Problem:-01

Use dynamic programming method to solve the following problem

Min $Z=y_1^2+y_2^2+y_3^2$

Subject to the constraints

$y_1+y_2+y_3 \geq 15$ and $y_1, y_2, y_3 \geq 0$

Solution:-

Since there are three decision variables y_1, y_2, y_3.

Therefore the given problem is to be solved in 3 stages.

Let $s_3= y_1+y_2+y_3$ ----------(1)

` $s_2= y_1+y_2$ $(=s_3-y_3)$ --------(2)

$s_1=y_1$ $(=s_2-y_2)$ ---------(3)

The functional relation is given by

$$f_1(s_1)=\min_{0 \leq y_1 \leq s_1} y_1^2 \qquad \text{--------(4)}$$

$$f_2(s_2)=\min_{0 \leq y_2 \leq s_2} y_1^2 + y_2^2 = \min_{0 \leq y_1, y_2 \leq s_2} \{f_1(s_1)+y_2^2\} \qquad \text{--------(5)}$$

$$f_3(s_3)=\min_{0\le y_3\le s_3} y_1^2+y_2^2+y_3^2=\min_{0\le y_3\le s_3}\{f_2(s_2)+y_3^2\} \quad \text{--------(6)}$$

To find the minimum value of $f_2(s_2)$

$f_2(s_2)=y_1^2+y_2^2$

$f_2(s_2)=(s_2-y_2)^2+y_2^2$ (equation (3))

Differentiate with respect to 'y_2' and assume equal to zero, we have

$2(s_2-y_2)(0-1)+2y_2=0$

$\Rightarrow y_2=s_2/2$ -------(7)

Sub the value of y_2 in $f_2(s_2)$, we have

$f_2(s_2)= y_1^2+y_2^2= y_1^2+(s_2/2)^2$ (by equation (5))

$f_2(s_2)= (s_2-y_2)^2+(s_2/2)^2$ (by equation (3))

$f_2(s_2)= (s_2- s_2/2)^2+(s_2/2)^2$

$f_2(s_2)= (s_2/2)^2+(s_2/2)^2=2s_2^2/4$

$f_2(s_2)=s_2^2/2.$ --------(8)

To find the minimum value of $f_3(s_3)$

$f_3(s_3)=f_2(s_2)+y_3^2$ (by equation (3))

$f_3(s_3)= s_2^2/2+y_3^2$ (by equation (8))

$f_3(s_3)= (s_3-y_3)^2/2+y_3^2$ (by equation (2))

Differentiate with respect to 'y_3' and equate it to zero, we have

$2(s_3-y_3)(0-1)/2+2y_3=0$

$2(15-y_3)(0-1)/2+2y_3=0$ (Sub $s_3=15$, data in given problem)

$(15-y_3)(-1)+2y_3=0$

$-15+y_3+2y_3=0$

$\Rightarrow y_3=5$ --------(9)

Substitute y_3 values in $f_3(s_3)$

$f_3(s_3)= (15-5)^2/2+5^2=100/2+25=75$

Optimal solution

Since $s_3=15$

(2)=>$s_2=s_3-y_3=15-5=10$ $f_2(s_2)=s_2^2/2=(10)^2/2$

(3)=>$s_1=s_2-y_2=10-5=5$ $(y_2=s_2/2)$

(3)=>$y_1^*=s_1=5$

(7)=>$y_2^*=s_2/2=10/2=5$

(9)=>$y_3^*=5$

Hence the optimal policy is (5,5,5) with $f_3^*(15)=75$.

Problem:-02

Use dynamic programming method to solve the following problem

Min $Z=y_1^2+y_2^2+y_3^2$

Subject to the constraints

$y_1+y_2+y_3=10$

and $y_1, y_2, y_3 \geq 0$

Solution:-

Since there are three decision variables y_1, y_2, y_3.

Therefore the given problem is 3 stage problem

Let $s_3= y_1+y_2+y_3$ ----------(1)

` $s_2= y_1+y_2$ $(=s_3-y_3)$ --------(2)

$s_1=y_1$ $(=s_2-y_2)$ ---------(3)

The functional relation is given by

$$f_1(s_1)=\min_{0 \leq y_1 \leq s_1} y_1^2 \quad \text{--------(4)}$$

$$f_2(s_2)=\min_{0 \leq y_2 \leq s_2} y_1^2 + y_2^2 = \min_{0 \leq y_2 \leq s_2} \{f_1(s_1) + y_2^2\} \quad \text{--------(5)}$$

$$f_3(s_3) = \min_{0 \le y_3 \le s_3} y_1^2 + y_2^2 + y_3^2 = \min_{0 \le y_3 \le s_3} \{f_2(s_2) + y_3^2\} \qquad \text{--------(6)}$$

To find the minimum value of $f_2(s_2)$

$f_2(s_2)=y_1^2+y_2^2$

$f_2(s_2)=(s_2-y_2)^2+y_2^2$ (equation (3))

Differentiate with respect to 'y_2' and assume equal to zero, we have

$2(s_2-y_2)(1-0)+2y_2=0$

$\Rightarrow y_2=s_2/2$ -------(7)

Sub the value of y_2 in $f_2(s_2)$, we have

$f_2(s_2)= y_1^2+y_2^2= y_1^2+(s_2/2)^2$ (by equation (5))

$f_2(s_2)= (s_2-y_2)^2+(s_2/2)^2$ (by equation (3))

$f_2(s_2)= (s_2- s_2/2)^2+(s_2/2)^2$

$f_2(s_2)= (s_2/2)^2+(s_2/2)^2=2s_2^2/4$

$f_2(s_2)=s_2^2/2.$ --------(8)

To find the minimum value of $f_3(s_3)$

$f_3(s_3)=f_2(s_2)+y_3^2$ (by equation (3))

$f_3(s_3)= s_2^2/2+y_3^2$ (by equation (8))

$f_3(s_3)= (s_3-y_3)^2/2+y_3^2$ (by equation (2))

Differentiate with respect to 'y_3' and equate it to zero, we have

$2(s_3-y_3)(0-1)/2+2y_3=0$

$2(10-y_3)(0-1)/2+2y_3=0$ (Sub $s_3=10$, data in given problem)

$(10-y_3)(-1)+2y_3=0$

$-10+y_3+2y_3=0$

$\Rightarrow y_3=10/3$ --------(9)

Substitute y_3 values in $f_3(s_3)$

$f_3(s_3)= (10-10/3)^2/2+(10/3)^2=(400/9)/2+100/9$

$f_3(s_3)=300/9=100/3$

Optimal solution

Since $s_3=10$

(2)=>$s_2=s_3-y_3=10-10/3=20/3$

(3)=>$s_1=s_2-y_2=20/3-10/3=10/3$ [$y_2=s_2/2$]

(3)=>$y_1^*=s_1=10/3$

(7)=>$y_2^*=s_2/2=(20/3)/2=10/3$

(9)=>$y_3^*=10/3$

Hence the optimal policy is (10/3, 10/3, 10/3) with $f_3^*(10/3)=100/3$.

EXCERCISE

1. Solve the following all integer programming problem using branch and bound method,

Max $Z=3x_1+2.5x_2,$

Subject to the conditions

$x_1+2x_2 \geq 20,$

$3x_1+2x_2 \geq 50$

and $x_1, x_2 \geq 0$ and integers. **[Ans. Sub problem D $x_1=14, x_2=4$ Z=52]**

2. Solve the following all integer programming problem using branch and bound method,

Max $Z=3x_1+5x_2,$

Subject to the conditions

$2x_1+4x_2 \leq 25,$

$x_1 \leq 8,$

$2x_2 \leq 10$

and $x_1, x_2 \geq 0$ and integers. **[Ans. Sub problem B $x_1=8, x_2=2$ Z=34]**

3. Consider the linear programming problem

Max $z=5x_1+4x_2$

Subject to constraints

$3x_1+4x_2 \leq 10$

and $x_1,x_2 \geq 0$ integers.

(a) Solve this model using branch and bound method

(b) Demonstrate the solution partition graphically.

4. Solve the following linear programming problem using branch and bound method.

Min $Z=3x_1+6x_2$

Subject to constraints

$7x_1+3x_2 \geq 40$

and $x_1,x_2 \geq 0$ and integers.

5. Solve the following linear programming problem using branch and bound method

Max $Z=100x_1+150x_2$

Subject to constraints

$8000x_1+4000x_2 \leq 40000$

$15x_1+30x_2 \leq 200$

and $x_1,x_2 \geq 0$ and integers. [Ans. x_1=1, x_2=6 and Max Z=1000]

6. Solve the following LP problem using Gomory's cutting plane algorithm

Max $Z=1.5x_1+3x_2+4x_3$

Subject to constraints

$2.5x_1+2x_2+4x_3 \leq 12$

$2x_1+4x_2-x_3 \leq 7$

and $x_1,x_2, x_3 \geq 0$ and integers.

7. Solve the following LP problem using cutting plane algorithm

Max $Z=2x_1+3x_2$

Subject to constraints

$x_1+3x_2 \leq 9$

$3x_1+x_2 \leq 7$

$x_1-x_2 \leq 1$

and $x_1,x_2 \geq 0$ and integers.

8. Solve the following LP problem using cutting plane algorithm

Max $Z=7x_1+6x_2$

Subject to constraints

$2x_1+3x_2 \leq 12$

$6x_1+5x_2 \leq 30$

and $x_1,x_2 \geq 0$ and integers.

9. Solve the following LP problem using cutting plane algorithm

Max $Z=5x_1+4x_2$

Subject to constraints

$x_1+x_2 \geq 2$

$5x_1+3x_2 \leq 15$

$3x_1+5x_2 \leq 15$

and $x_1,x_2 \geq 0$ and integers.

10. Solve the following LP problem using cutting plane algorithm

Max $Z=-3x_1+x_2+3x_3$

Subject to constraints

$-x_1+2x_2+x_3 \leq 4$

$2x_1-1.5x_3 \leq 1$

$x_1-3x_2+2x_3 \leq 3$

and $x_1, x_2 \geq 0$ and x_3 is non negative integer.

11. Solve the following LP problem using cutting plane algorithm

Max $Z=x_1+x_2$

Subject to constraints

$2x_1+5x_2 \geq 16$

$6x_1+5x_2 \leq 30$

and $x_2 \geq 0$ and x_1 is non negative integer.

12. Solve the following LP problem using cutting plane algorithm

Min $Z=4x_1+3x_2+5x_3$

Subject to constraints

$2x_1-2x_2+4x_3 \geq 7$

$2x_1+6x_2-2x_3 \geq 5$

$x_1-3x_2+2x_3 \leq 3$

and $x_2 \geq 0$ and x_1, x_3 are non negative integers.

13. Solve the following LP problem using cutting plane algorithm

Max $Z=110x_1+100x_2$

Subject to constraints

$6x_1+5x_2 \leq 29$

$4x_1+14x_2 \leq 48$

and $x_1, x_2 \geq 0$ and integers.

LECTURE NOTES

OPERATIONS RESEARCH

UNIT-II

LPP STANDARD FORM I

Max Z=CX

Subject to constraints

AX=b, and

$X \geq 0$

If the objective function is considered as one of the constraints equation then the LP problem in its standard form is written by *Z-CX=0 and AX=b,* $X \geq 0$

Problem:-01

Use the revised simplex method to solve the following LP problem

Max $z=3x_1+5x_2$

Subject to

$x_1 \leq 4$

$x_2 \leq 6$

$3x_1+2x_2 \leq 18$

and $x_1, x_2 \geq 0$

Solution:-

The standard form I corresponding to the given problem is written by

$z-3x_1-5x_2=0$ (Add objective function as constraint)

$x_1+s_1=4$ (Add slack variable $s_1 \geq 0$)

$x_2+s_2=6$ (Add slack variable $s_2 \geq 0$)

$3x_1+2x_2+s_3=18$ (Add slack variable $s_3 \geq 0$)

and $x_1, x_2, s_1, s_2, s_3 \geq 0$

Initial basic feasible solution is $s_1=4$, $s_2=6$, $s_3=18$ and Max Z=0 ($x_1=0$, $x_2=0$)

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_2^{(1)}}$
	Z	s_1	s_2	s_3	C_k-z_k	x_1	x_2
z=0	1	0	0	0		-3	-5
$s_1=4$	0	1	0	0		1	0
$s_2=6$	0	0	1	0		0	1
$s_3=18$	0	0	0	1		3	2

$$C_k-z_k = \text{Max}\left\{-\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}-3 & -5\\ 1 & 0\\ 0 & 1\\ 3 & 2\end{pmatrix}\right\} = \text{Max}\{-(-3,-5)\} = \text{Max}\{3,5\} = 5,$$

Corresponding to x_2 (k=2)

$$y_k^{(1)} = B^{(1)} a_k^{(1)} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-5\\ 0\\ 1\\ 2\end{pmatrix} = \begin{pmatrix}-5\\ 0\\ 1\\ 2\end{pmatrix}$$

x_2 enters the basis

	z	s_1	s_2	s_3	$y_2^{(1)}$	Min X_B/y_{rj}
z=0	1	0	0	0	-5	-
s_1=4	0	1	0	0	0	-
s_2=6	0	0	1	0	[1]	6
s_3=18	0	0	0	1	2	9

s_2 leaves the basis

First row values for new table

0-(-5)(6)/1=30, 1-(-5)(0)/1=1, 0-(-5)(0)/1=0,

0-(-5)(1)/1=5, 0-(0)(-5)/1=0

Second row values for new table

4-(0)(6)/1=4, 0-(0)(0)/1=0, 1-(0)(0)/1=1,

0-(0)(1)/1=0, 0-(0)(0)/1=0,

Third row values for new table

6/1, 0/1, 0/1, 1/1, 0/1

Fourth row values for new table

18-(6)(2)/1=6, 0-(0)(2)/1=0, 0-(0)(2)/1=0,

0-(1)(2)/1=-2, 1-(0)(2)/2=1

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		**$a_1^{(1)}$**	**$a_3^{(1)}$**
	Z	s_1	x_2	s_3	C_k-z_k	x_1	s_2
z=30	1	0	5	0		-3	0
s_1=4	0	1	0	0		1	0
x_2=6	0	0	1	0		0	1
s_3=6	0	0	-2	1		3	0

$$C_k\text{-}z_k = \text{Max}\left\{-\begin{pmatrix}1 & 0 & 5 & 0\end{pmatrix}\begin{pmatrix}-3 & 0\\ 1 & 0\\ 0 & 1\\ 3 & 0\end{pmatrix}\right\} = \text{Max}\{-(-3,5)\} = \text{Max}\{3,-5\} = 3,$$

Corresponding to x_1 (k=1)

$$y_k^{(1)} = B^{(1)} a_k^{(1)} = \begin{pmatrix}1 & 0 & 5 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -2 & 1\end{pmatrix}\begin{pmatrix}-3\\ 1\\ 0\\ 3\end{pmatrix} = \begin{pmatrix}-3\\ 1\\ 0\\ 3\end{pmatrix}$$

x_1 enters the basis

	z	s_1	x_2	s_3	$y_1^{(1)}$	Min X_B/y_{rj}
z=30	1	0	5	0	-3	-
s_1=4	0	1	0	0	1	4
x_2=6	0	0	1	0	0	-
s_3=6	0	0	-2	1	[3]	2

s_3 leaves the basis

First row for new table

30-(-3)(6)/3=36, 1-(0)(-3)/3=1, 0-(0)(-3)/3=0,

5-(-2)(-3)/3=3, 0-(1)(-3)/3=1

Second row for new table

4-(1)(6)/3=2, 0-(0)(1)=0, 1-(0)(1)/3=1,

0-(-2)(1)/3=2/3, 0-(1)(1)/3=-1/3

Third row for new table

6-(6)(0)/3=6, 0-(0)(0)/3=0, 0-(0)(0)/3=0,

1-(-2)(0)/3=1, 0-(1)(0)/3=0

Fourth row for new table

6/3=2, 0/3=0, 0/3=0, -2/3, 1/3, 3/3=1,

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_3^{(1)}}$
	Z	s_1	x_2	x_1	C_k-z_k	s_3	s_2
z=36	1	0	3	1		0	0
s_1=2	0	1	2/3	-1/3		0	0
x_2=6	0	0	1	0		0	1
x_1=2	0	0	-2/3	1/3		1	0

$$C_k\text{-}z_k=\text{Max}\left\{-\begin{pmatrix}1 & 0 & 3 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}\right\}=\text{Max}\{-(1,3)\}=\text{Max}\{-1,-3\},$$

Since C_k-$z_k \leq 0$ for all k, therefore the above table is optimal table,

Hence the optimum solution is **x_1=2, x_2=6 and Max Z=36**

Problem:-02

Use the revised simplex method to solve the following LP problem

Max $z=2x_1+x_2$

Subject to

$3x_1+4x_2 \leq 6$

$6x_1+x_2 \leq 3$

and $x_1, x_2 \geq 0$

Solution:-

The standard form I corresponding to the given problem is written by

$z-2x_1-x_2=0$ (Add objective function as constraint)

$3x_1+4x_2+s_1=6$ (Add slack variable $s_1 \geq 0$)

$6x_1+x_2+s_2=3$ (Add slack variable $s_2 \geq 0$)

and $x_1, x_2, s_1, s_2 \geq 0$

Initial basic feasible solution is $s_1=6$, $s_2=3$ and Max Z=0 ($x_1=0$, $x_2=0$)

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_2^{(1)}}$
	Z	s_1	s_2	C_k-Z_k	x_1	x_2
$z=0$	1	0	0		-2	-1
$s_1=6$	0	1	0		3	4
$s_2=3$	0	0	1		6	1

$C_k\text{-}z_k$=Max $\left\{-\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}-2 & -1\\ 3 & 4\\ 6 & 1\end{pmatrix}\right\}$=Max{-(-2,-1)}=Max {2,1}=2,

Corresponding to x_1 (k=1)

$y_k^{(1)}=B^{(1)}a_k^{(1)}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-2\\ 3\\ 6\end{pmatrix}=\begin{pmatrix}-2\\ 3\\ 6\end{pmatrix}$

x_1 enters the basis

	z	s_1	s_2	$y_1^{(1)}$	Min X_B/y_{rj}
z=0	1	0	0	-2	-
s_1=6	0	1	0	3	2
s_2=3	0	0	1	[6]	0.5

First row values for new table

0-(-2)(3)/6=1, 1-(-2)(0)/1=1, 0-(-2)(0)/1=0,

0-(-2)(1)/6=1/3,

Second row values for new table

6-(3)(3)/6=9/2, 0-(0)(3)/6=0, 1-(0)(3)/6=1,

0-(3)(1)/6=-1/2,

Third row values for new table

3/6, 0/6, 0/6, 6/6,

s_2 leaves the basis

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_3^{(1)}}$
	Z	s_1	x_1	C_k-Z_k	s_2	x_2
z=1	1	0	1/3		0	-1
s_1=9/2	0	1	-1/2		0	4
x_1=3/6	0	0	1/6		1	1

C_k-z_k=Max $\left\{ -\begin{pmatrix} 1 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 4 \\ 1 & 1 \end{pmatrix} \right\}$ =Max{-(1/3,-2/3)}=Max {-1/3,2/3}=2/3,

Corresponding to x_2 (k=2)

$y_k^{(1)}=B^{(1)}a_k^{(1)}=\begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/6 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}=\begin{pmatrix} -2/3 \\ 7/2 \\ 1/6 \end{pmatrix}$

x_2 enters the basis

	z	s_1	x_1	*$y_2^{(1)}$*	Min X_B/y_{rj}	
z=1	1	0	1/3	-2/3	-	
s_1=9/2	0	1	-1/2	[7/2]	9/7	←
x_1=1/2	0	0	1/6	1/6	3	
				↑		

s_1 leaves the basis

First row for new table

1-(-2/3)(9/2)/(7/2)=13/7, 1-(0)(-2/3)/(7/2)=1,

0-(1)(-2/3)/(7/2)=4/21, 1/3-(-2/3)(-1/2)/(7/2)=5/21

Second row for new table

(9/2)/(7/2)=9/7, 0/(7/2)=0, 1/(7/2)=2/7,

(-1/2)/(7/2)=-1/7, (7/2)/(7/2)=1

Third row for new table

1/2-(9/2)(1/6)/(7/2)=2/7, 0-(0)(1/6)/(7/2)=0,

0-(1)(1/6)/(7/2)=-1/21, 1/6-(-1/2)(1/6)/(7/2)=4/21

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_3^{(1)}}$
	Z	x_2	x_1	C_k-z_k	s_2	s_1
z=13/7	1	4/21	5/21		0	0
x_2=9/7	0	2/7	-1/7		0	1
x_1=2/7	0	-1/21	4/21		1	0

C_k-z_k=Max $\left\{-\begin{pmatrix}1 & 4/21 & 5/21\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix}\right\}=$ Max{-(5/21,4/21)}=Max{-5/21,-4/21},

Since C_k-$z_k \leq 0$ for all k, therefore the above table is optimal table,

Hence the optimum solution is **x_1=2/7, x_2=9/7 and Max Z=13/7.**

Problem:-03

Use the revised simplex method to solve the following LP problem

Max $z=x_1+x_2+3x_3$

Subject to

$3x_1+2x_2+x_3 \leq 3$

$2x_1+x_2+2x_3 \leq 2$

and $x_1,x_2,x_3 \geq 0$

Solution:-

The standard form I corresponding to the given problem is written by

$z-x_1-x_2-3x_3=0$ (Add objective function as constraint)

$3x_1+2x_2+x_3+s_1=3$ (Add slack variable $s_1 \geq 0$)

$2x_1+x_2+2x_3+s_2=2$ (Add slack variable $s_2 \geq 0$)

and $x_1, x_2, x_3, s_1, s_2 \geq 0$

Initial basic feasible solution is $s_1=3$, $s_2=2$ and Max Z=0 ($x_1=0$, $x_2=0$, $x_3=0$)

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_2^{(1)}}$	$\mathbf{a_3^{(1)}}$
	Z	s_1	s_2	C_k-Z_k	x_1	x_2	x_3
z=0	1	0	0		-1	-1	-3
$s_1=3$	0	1	0		3	2	1
$s_2=2$	0	0	1		2	1	2

$C_k-Z_k=\text{Max}\left\{-\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}-1 & -1 & -3\\ 3 & 2 & 1\\ 2 & 1 & 2\end{pmatrix}\right\}$=Max{-(-1,-1,-3)}=Max {1,1,3}=3,

Corresponding to x_3 (k=3)

$y_k^{(1)}=B^{(1)}a_k^{(1)}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-3\\ 1\\ 2\end{pmatrix}=\begin{pmatrix}-3\\ 1\\ 2\end{pmatrix}$

x_3 enters the basis

	Z	s_1	s_2	$y_3^{(1)}$	Min X_B/y_{rj}
z=0	1	0	0	-3	-
s_1=3	0	1	0	1	3
s_2=2	0	0	1	[2]	1 ←

↑

First row values for new table

0-(2)(-3)/2=3, 1-(-3)(0)/2=1, 0-(-3)(0)/2=0,

0-(-3)(1)/2=3/2,

Second row values for new table

3-(1)(2)/2=2, 0-(0)(1)/2=0, 1-(0)(1)/2=1,

0-(1)(1)/2=-1/2,

Third row values for new table

2/2=1, 0/2=0, 0/2=0, 1/2,

s_2 leaves the basis

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_3^{(1)}}$	
	Z	s_1	x_1	C_k-z_k	x_1	x_2	s_2
z=3	1	0	3/2		-1	-1	0
s_1=2	0	1	-1/2		3	2	0
x_3=1	0	0	1/2		2	1	1

$$C_k\text{-}z_k=\text{Max}\left\{-\begin{pmatrix}1 & 0 & 3/2\end{pmatrix}\begin{pmatrix}-1 & -1 & 0\\ 3 & 2 & 0\\ 2 & 1 & 1\end{pmatrix}\right\}=\text{Max}\{-(2, 1/2, 3/2)\}$$

=Max {-2, -1/2, -3/2}

Since C_k-$z_k \leq 0$ for all k, therefore the above table is optimal table,

Hence the optimum solution is **x_1=0, x_2=0, x_3=1 and Max Z=3.**

Problem:-04

Use the revised simplex method to solve the following LP problem

Max $z=x_1+2x_2$

Subject to

$x_1+x_2 \leq 3$

$x_1+2x_2 \leq 5$

$3x_1+x_2 \leq 6$

and $x_1,x_2 \geq 0$

Solution:-

The standard form I corresponding to the given problem is written by

$z-x_1-2x_2=0$ (Add objective function as constraint)

$x_1+x_2+s_1=3$ (Add slack variable $s_1 \geq 0$)

$x_1+2x_2+s_2=5$ (Add slack variable $s_2 \geq 0$)

$3x_1+x_2+s_3=6$ (Add slack variable $s_3 \geq 0$)

and $x_1,x_2, s_1, s_2, s_3 \geq 0$

Initial basic feasible solution is $s_1=3$, $s_2=5$, $s_3=6$ and Max Z=0 ($x_1=0$, $x_2=0$)

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		$a_1^{(1)}$	$a_2^{(1)}$
	Z	s_1	s_2	s_3	C_k-Z_k	x_1	x_2
z=0	1	0	0	0		-1	-2
$s_1=3$	0	1	0	0		1	1
$s_2=5$	0	0	1	0		1	2
$s_3=6$	0	0	0	1		3	1

$$C_k - z_k = \text{Max}\left\{-\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}-1 & -2\\ 1 & 1\\ 1 & 2\\ 3 & 1\end{pmatrix}\right\} = \text{Max}\{-(-1,-2)\} = \text{Max}\{1,2\} = 2,$$

Corresponding to x_2 (k=2)

$$y_k^{(1)} = B^{(1)} a_k^{(1)} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-2\\ 1\\ 2\\ 1\end{pmatrix} = \begin{pmatrix}-2\\ 1\\ 2\\ 1\end{pmatrix}$$

x_2 enters the basis

	z	s_1	s_2	s_3	$y_2^{(1)}$	Min X_B/y_{rj}
z=0	1	0	0	0	-2	
s_1=3	0	1	0	0	1	3
s_2=5	0	0	1	0	[2]	5/2
s_3=6	0	0	0	1	1	6

s_2 leaves the basis

First row values for new table

0-(5)(-2)/2=5, 1-(-2)(0)/2=1, 0-(-2)(0)/2=0,

0-(-2)(1)/2=1, 0-(0)(-2)/2=0

Second row values for new table

3-(1)(5)/2=1/2, 0-(1)(0)/2=0, 1-(0)(1)/2=1,

0-(1)(1)/2=-1/2, 0-(0)(1)/2=0,

Third row values for new table

5/2, 0/2=0, 0/2=0, 1/2, 0/2=0

Fourth row values for new table

6-(5)(1)/2=7/2, 0-(0)(1)/2=0, 0-(0)(1)/2=0,

0-(1)(1)/2=-1/2, 1-(0)(1)/2=1

	$\beta_0^{(1)}$	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$		$\mathbf{a_1^{(1)}}$	$\mathbf{a_3^{(1)}}$
	Z	s_1	x_2	s_3	C_k-z_k	x_1	s_2
z=5	1	0	1	0		-1	0
s_1=1/2	0	1	-1/2	0		1	0
x_2=5/2	0	0	1/2	0		1	1
s_3=7/2	0	0	-1/2	1		3	0

$$C_k\text{-}z_k=\text{Max}\left\{-\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 1 & 0\\ 1 & 1\\ 3 & 0\end{pmatrix}\right\}=\text{Max}\{-(0,1)\}=\text{Max}\{-1\}$$

Since C_k-$z_k \leq 0$ for all k, therefore the above table is optimal table,

Hence the optimum solution is **x_1=0, x_2=5/2 and Max Z=5.**

Procedure

Step:-01

Verify whether all variables has 0 lower bound, if not suppose x_k has a positive lower bound, then make its lower bound zero using substitution $x_k'=x_k$-LB, where LB is lower bound of x_k. Update x_k' where ever we have x_k by using the substitution.

Step:-02

The objective function is should be maximisation type , otherwise change the given minimisation problem into maximisation type problem.

Step:-03

Solve using usual simplex method, add upper bound (u_r) of variables ($0 \leq x_r \leq u_r$) on the top of the simplex table, for slack or surplus variables upper bound is not defined, therefore it is assumed to be infinity ∞. If the current solution is not optimal proceed to step 4. F

Step:-04

Find the initial basic feasible solution, then select a non-basic variable to enter (using simple procedure) and the basic variable to leave (using the following procedure)

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if \ y_{ir} > 0\right\}$$

Here y_{ir} are the values in the r^{th} column which is going to enter into the basis.

$$\theta_2 = \min\left\{\frac{u_r - X_{Bi}}{y_{ir}}, if \ y_{ir} < 0\right\}$$

Here u_r is the upper bound value of the r^{th} column which is going to enter into the basis.

$\theta = \min\{\theta_1, \theta_2, u_3\}$, *Go to Step 05*

Step:-05

(i) If $\theta = \theta_1$ then corresponding basic variable is removed from the basis.

Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*

(ii) If $\theta = \theta_2$ then corresponding basic variable is removed from basis,

Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*

The following changes to be done in the updated table

(a) Change of leaving basic variables column values

We have to put of leaving basic variable x_r at its upper bound use the equation $x_r=u_r-x_r'$, $0 \leq x_r' \leq u_r$. where x_r' is new variable to be written in place of x_r, and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y=X_B$ column

$(x_{BK})_r=(x_{Bk})'-y_{kr}u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.

Write the updated table. *Go to Step 06*.

(iii) If $\theta = u_r$ then variable x_r is not leaving the basis, but given its upper bound value is calculated as $x_r=u_r-x_r'$, $0 \leq x_r' \leq u_r$

The new updated table is same as old one, but with the following two column changes

(a) x_r Column

Replace x_r by x_r', and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y=X_B$ column

$(x_{BK})_r=(x_{Bk})'-y_{kr}u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.

Write the updated table. *Go to Step 06*.

Step:-06

After getting updated table, if it is optimal then stop calculation, otherwise use step:-04 for new table.

Problem:-01

Use bounded variable method to solve the following LP problem

Max $z=x_2+3x_3$

Subject to

$x_1+x_2+x_3 \leq 10$

$x_1-2x_3 \geq 0$

$2x_1-x_3 \leq 10$

and $0 \leq x_1 \leq 8$, $0 \leq x_2 \leq 4$, $x_3 \geq 0$

Solution:-

The standard form of given LP problem is

Max $z= x_2+3x_3+0s_1+0s_2+0s_3$

$x_1+x_2+x_3+s_1= 10$ (Add slack variable $s_1 \geq 0$)

$-x_1+2x_2+s_2=0$ (Add slack variable $s_2 \geq 0$)(x by -1)

$2x_1-x_3+s_3=10$ (Add slack variable $s_3 \geq 0$)

and $0 \leq x_1 \leq 8$, $0 \leq x_2 \leq 4$, $0 \leq x_3 < \infty$, $0 \leq s_1 < \infty$, $0 \leq s_2 < \infty$, $0 \leq s_3 < \infty$

Initial basic feasible solution is *$s_1=10$, $s_2=0$, $s_3=10$, $x_1=0$, $x_2=0$, $x_3=0$ and Max $z=0$*

	u_j	8	4	∞	∞	∞	∞
	c_j	0	1	3	0	0	0
CB	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	$s_1=10$	1	1	1	1	0	0
0	$s_2=0$	-1	0	[2]	0	1	0
0	$s_3=10$	2	0	-1	0	0	1
	z_j	0	0	0	0	0	0
	z_j-c_j	0	-1	-3	0	0	0

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_3 enters the basis

To find the variable leaves the basis

$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if\ \ y_{ir} > 0\right\}$ [y_{ir} are the selected column values]

$\theta_1 = \min\left\{\frac{10}{1}, \frac{0}{2}\right\}$

$\theta_1 = 0$ [corresponding to the variable s_2]

$\theta_2 = \min\left\{\frac{u_r - X_{Bi}}{y_{ir}}, if\ \ y_{ir} < 0\right\}$

$\theta_2 = \min\left\{\frac{u_3 - X_{B3}}{-y_{33}}\right\} = \min\left\{\frac{\infty - 10}{-(-1)}\right\} = \infty$ [corresponding to the variable x_3]

$\theta = \min\{\theta_1, \theta_2, u_3\} = \min\{0, \infty, \infty\} = 0$ [corresponding to the variable s_2]

$\theta = 0 = \theta_1$ [corresponding to the variable s_2]

[Rule :-

If $\theta = \theta_1$ then corresponding basic variable is removed from the basis. Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*]

s_2 leave the basis

First row values

10-(0)(1)/2=10	1-(-1)(1)/2=3/2	1-(0)(1)/2=1
1-(0)(1)/2=1	0-(1)(1)/2=-1/2	0-(0)(1)/2=0

Second row values

0/2=0 -1/2 0/2=0 2/2=1 0/2=0 1/2 0/2=0

Third row values

10-(0)(-1)/2=10	2-(-1)(-1)/2=3/2	0-(0)(-1)/2=0
0-(0)(-1)/2=0	0-(1)(-1)/2=1/2	1-(0)(-1)/2=1

	u_j	8	4	∞	∞	∞	∞
	c_j	0	1	3	0	0	0
CB	$Y=X_B$	x_1	x_2	x_3	s_1	s_2	s_3
0	$s_1=10$	[3/2]	1	0	1	-1/2	0
-3	$x_3=0$	-1/2	0	1	0	1/2	0
0	$s_3=10$	3/2	0	0	0	1/2	1
	z_j	-3/2	0	3	0	3/2	0
	z_j-c_j	-3/2	-1	0	0	3/2	0

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_1 enters the basis

To find the variable leaves the basis

$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if\ \ y_{ir} > 0\right\}$ [y_{ir} are the values of selected column]

$\theta_1 = \min\left\{\frac{20}{3}, \frac{20}{3}\right\}$

$\theta_1 = 20/3$ [corresponding to the variable s_1, s_3]

$\theta_2 = \min\left\{\frac{u_r - X_{Bi}}{y_{ir}}, if\ \ y_{ir} < 0\right\}$

$\theta_2 = \min\left\{\frac{u_1 - X_{B1}}{-y_{21}}\right\} = \min\left\{\frac{8-0}{-(-1/2)}\right\} = 16$ [corresponding to the variable x_3]

$\theta = \min\{\theta_1, \theta_2, u_1\} = \min\{20/3,\ \ 16,\ \ 8\} = 20/3$ [corresponding to variables s_1, s_3]

$\theta = 20/3 = \theta_1$

[Rule :-

If $\theta = \theta_1$ then corresponding basic variable is removed from the basis. Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*]

s_1 leaves the basis

First row values

10/(3/2)=20/3 (3/2)/(3/2)=1 1/(3/2)=2/3 0/(3/2)=0

1/(3/2)=2/3 (-1/2)/(3/2)=-1/3 0/(2/3)=0

Second row values

0-(10)(-1/2)/(3/2)=10/3 0-(1)(-1/2)/(3/2)=1/3 1-(0)(-1/2)/(3/2)=1

0-(1)(-1/2)(3/2)=1/3 1/2-(-1/2)(-1/2)(3/2)=1/3 0-(0)(-1/2)/(3/2)=0

Third row values

10-(10)(3/2)/(3/2)=0 0-(1)(3/2)/(3/2)=-1 0-(0)(3/2)/(3/2)=0

0-(1)(3/2)/(3/2)=-1 1/2-(-1/2)(3/2)/(3/2)=1 1-(0)(3/2)/(3/2)=1

	u_j	8	4	∞	∞	∞	∞
	c_j	0	1	3	0	0	0
CB	X_B	x_1	x_2	x_3	s_1	s_2	s_3
0	x_1=20/3	1	2/3	0	2/3	-1/3	0
-3	x_3=10/3	0	1/3	1	1/3	1/3	0
0	s_3=0	0	-1	0	-1	1	1
	z_j	0	1	3	1	1	0
	z_j-c_j	0	0	0	1	1	0

Since all $z_j-c_j \geq 0$, therefore the above table is optimal.

Hence the optimal solution is **x_1=0, x_2=0 , x_3=10/3 and Max z=10**

Problem:-02

Use bounded variable method to solve the following LP problem

Max $z=x_1+3x_2-2x_3$

Subject to

$x_2-2x_3 \leq 1$

$2x_1+x_2+2x_3 \leq 8$

and $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 3$, $0 \leq x_3 \leq 2$

Solution:-

The standard form of given LP problem is

Max $z= x_1+3x_2-2x_3+0s_1+0s_2+0s_3$

$x_2-2x_3+s_1= 1$ (Add slack variable $s_1 \geq 0$)

$2x_1+x_2+2x_3+s_2=8$ (Add slack variable $s_2 \geq 0$)

and $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 3$, $0 \leq x_3 \leq 2$, $0 \leq s_1 \leq \infty$, $0 \leq s_2 \leq \infty$.

Initial basic feasible solution is *$s_1=1$, $s_2=8$, $x_1=0$, $x_2=0$, $x_3=0$ and Max $z=0$*

	u_j	1	3	2	∞	∞	
	c_j	1	3	-2	0	0	
CB	X_B	x_1	x_2	x_3	s_1	s_2	
0	$s_1=1$	0	[1]	-2	1	0	←
0	$s_2=8$	2	1	2	0	1	
	z_j	0	0	0	0	0	
	z_j-c_j	-1	-3	2	0	0	
			↑				

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_2 enters the basis

To find the variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if \ y_{ir} > 0\right\}$$

$\theta_1 = \min\{1,8\} = 1$ [corresponding to the variable s_1]

$$\theta_2 = \min\left\{\frac{u_r - X_{Bi}}{y_{ir}}, if \;\; y_{ir} < 0\right\}$$

$\theta_2 = \infty$

$\theta = \min\{\theta_1, \theta_2, u_2\} = \min\{1, \infty, 3\} = 1$ [corresponding to the variable s_1]

$\theta = 1 = \theta_1$

[Rule :-

If $\theta = \theta_1$ then corresponding basic variable is removed from the basis. Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*]

s_1 leave the basis

First row values

1/1=1 0/1=0 1/1=1 -2/1=-2 1/1=1 0/1=0

Second row values

8-(1)(1)/1=7 2-(0)(1)/1=2 2-(-2)(1)/1=4 0-(1)(1)/1=-1 1-(1)(0)/1=1

	u_j	1	3	2	∞	∞
	c_j	1	3	-2	0	0
CB	X_B	x_1	x_2	x_3	s_1	s_2
3	x_2=1	0	1	[-2]	1	0
0	s_2=7	2	0	4	-1	1
	z_j	0	3	-6	3	0
	z_j-c_j	-1	0	-4	3	0

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_3 enters the basis

To find the variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if\ y_{ir} > 0\right\}$$

$$\theta_1 = \min\left\{\frac{7}{4}\right\} = 7/4$$ [corresponding to the variable s_2]

$$\theta_2 = \min\left\{\frac{u_i - X_{Bi}}{a_{ir}}, if\ a_{ir} < 0\right\}$$

$$\theta_2 = \min\left\{\frac{u_3 - X_{B1}}{-y_{13}}\right\} = \min\left\{\frac{2-1}{-(-2)}\right\} = \min\{1/2\} = 1/2$$

[corresponding to the variable x_2]

$$\theta = \min\{\theta_1, \theta_2, u_3\} = \min\{7/4,\quad 1/2,\quad 2\} = 1/2$$

$$\theta = 1/2 = \theta_2$$ [corresponding to the variable x_2]

x_2 leaves the basis

[If $\theta = \theta_2$ then corresponding basic variable is removed from basis,

Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table.

The following changes to be done in the updated table

(a) Change of leaving basic variables column values

We have to put of leaving basic variable x_r at its upper bound use the equation $x_r = u_r - x_r'$, $0 \leq x_r' \leq u_r$. where x_r' is new variable to be written in place of x_r, and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y=X_B$ column

$(x_{BK})_r = (x_{Bk})' - y_{kr}u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.]

First row values

1/(-2)=-1/2   0/(-2)=0   1/(-2)=-1/2   (-2)/(-2)=1

1/(-2)=-1/2   0/(-2)=0

Second row values

7-(1)(4)/(-2)=9   2-(0)(4)/(-2)=2   0-(1)(4)/(-2)=2   0   -1-(4)(1)/(-2)=1

1-(0)(4)/(-2)=1

	u_j	1	3	2	∞	∞
	c_j	1	3	-2	0	0
CB	X_B	x_1	x_2	x_3	s_1	s_2
-2	x_3=-1/2	0	-1/2	1	-1/2	0
0	s_2=9	2	2	0	1	1
	(z_0=1) z_j	0	1	-2	1	0
	z_j-c_j	-1	-2	0	1	0

The following must updated in the above table

Let $x_2=u_2-x_2'=3-x_2'$, then

$0 \le x_2 \le 3$

$0 \le 3 - x_2' \le 3$

$0-3 \le 3 - x_2' - 3 \le 3-3$ (subtract 3)

$-3 \le -x_2' \le 0$

$3 \ge x_2' \ge 0$ (x by -1)

i.e $0 \le x_2' \le 3$

We know that

$x_{Bk}=x_{Bk}'-a_{kr}u_r$

$x_{B1}=x_{B1}'-y_{12}u_2$=-1/2-(-1/2)(3)=1

$x_{B2}=x_{B2}'-y_{22}u_2$=9-(2)(3)=3

$z_0 = z_0' - (z_2 - c_2)u_2 = 1 - (-2)(3) = 7$

	u_j	1	3	2	∞	∞
	c_j	1	-3	-2	0	0
CB	X_B	x_1	x_2'	x_3	s_1	s_2
-2	$x_3=1$	0	1/2	1	-1/2	0
0	$s_2=3$	2	-2	0	1	1
	$(z_0=7)$ z_j	0	-1	-2	1	0
	z_j-c_j	-1	+2	0	1	0

↑

x_1 enters the basis

To find the basic variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if \ y_{ir} > 0\right\}$$

$$\theta_1 = \min\{3/2\} = 1.5$$ [corresponding to the variable s_2]

$$\theta_2 = \min\left\{\frac{u_i - X_{Bi}}{y_{ir}}, if \ y_{ir} < 0\right\}$$

$$\theta_2 = \infty$$

$$\theta = \min\{\theta_1, \theta_2, u_1\} = \min\{1.5, \quad \infty, \quad 1\} = 1$$

$$\theta = 1 = u_1$$

Rule:-

If $\theta = u_r$ then variable x_r is not leaving the basis, but its upper bound value is calculated as $x_r = u_r - x_r'$, $0 \le x_r' \le u_r$

The new updated table is same as old one, but with the following two column changes

(a) x_r Column

Replace x_r by x_r', and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y=X_B$ column

$(x_{BK})_r=(x_{Bk})'-y_{kr}u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.]

Let $x_1=u_1-x_1'=1-x_1'$, then

$0 \le x_1 \le 1$

$0 \le 1-x_1' \le 1$

$0-1 \le 1-x_1'-1 \le 1-1$ (subtract 1)

$-1 \le -x_1' \le 0$

$1 \ge x_1' \ge 0$ (x by -1)

i.e $0 \le x_1' \le 1$

We know that

$x_{Bk}=x_{Bk}-a_{kr}u_r$

$x_{B1}=x_{B1}-y_{11}u_1=1-(0)(1)=1$

$x_{B2}=x_{B2}-y_{21}u_1=3-(2)(1)=1$

$z_0=z_0'-(z_1-c_1)u_1=7-(-1)(1)=8$

	u_j	1	3	2	∞	∞
	c_j	-1	-3	-2	0	0
CB	X_B	x_1'	x_2'	x_3	s_1	s_2
-2	$x_3=1$	0	1/2	1	-1/2	0
0	$s_2=1$	-2	2	0	1	1
	$(z_0=8)$ z_j	0	-1	-2	1	0
	z_j-c_j	1	2	0	1	0

Since all $z_j-c_j \ge 0$, therefore the above table is optimal

$x_1'=0$

$1-x_1=0$ [since $x_1=1-x_1'$]

i.e $x_1=1$

$x_2'=0$ [since $x_2=3-x_2'$]

$3-x_2=0$

i.e $x_2=3$

$x_3=1$

Hence the optimal solution is **$x_1=1$, $x_2=3$, $x_3=1$ and**

Max $z=1(1)+3(3)-2(1)=8$

Problem:-03

Use upper bound algorithm to solve

Max $z=3x_1+5x_2+2x_3$

Subject to

$x_1+x_2+2x_3 \leq 14$

$2x_1+4x_2+3x_3 \leq 34$

and $0 \leq x_1 \leq 4$, $0 \leq x_2 \leq 10$, $0 \leq x_3 \leq 3$

Solution:-

The standard form of given LP problem is

Max $z= 3x_1+5x_2+2x_3+0s_1+0s_2+0s_3$

$x_1+x_2+2x_3+s_1= 1$ (Add slack variable $s_1 \geq 0$)

$2x_1+4x_2+3x_3+s_2=8$ (Add slack variable $s_2 \geq 0$)

and $0 \leq x_1 \leq 4$, $0 \leq x_2 \leq 10$, $0 \leq x_3 \leq 3$, $0 \leq s_1 \leq \infty$, $0 \leq s_2 \leq \infty$.

Initial basic feasible solution is $s_1=14$, $s_2=34$, $x_1=0$, $x_2=0$, $x_3=0$ and

Max $z=0$

	u_j	4	10	3	∞	∞	
	c_j	3	5	2	0	0	
CB	X_B	x_1	x_2	x_3	s_1	s_2	
0	$s_1=14$	1	1	2	1	0	
0	$s_2=34$	2	[4]	3	0	1	←
	z_j	0	0	0	0	0	
	z_j-c_j	-3	-5	-2	0	0	
			↑				

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_2 enters the basis

To find the variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if\ \ y_{ir} > 0\right\}$$

$\theta_1 = \min\{14/1, \quad 34/4\} = 34/4 = 8.5$ [corresponding to the variable s_2]

$$\theta_2 = \min\left\{\frac{u_r - X_{Bi}}{y_{ir}}, if\ \ y_{ir} < 0\right\}$$

$\theta_2 = \infty$

$\theta = \min\{\theta_1, \theta_2, u_2\} = \min\{8.5, \infty, 10\} = 8.5$

$\theta = 8.5 = \theta_1$ [corresponding to the variable s_2]

[Rule :-

If $\theta = \theta_1$ then corresponding basic variable is removed from the basis. Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*]

s_2 leave the basis

First row values

14-(34)(1)/4=11/2 1-(2)(1)/4=1/2 2-(3)(1)/4=5/4

1-(0)(1)/4=1 0-(1)(1)/4=-1/4

Second row values

34/4=17/2 2/4=1/2 4/4=1 3/4 0/4=0 1/4

	u_j	4	10	3	∞	∞
	c_j	3	5	2	0	0
CB	X_B	x_1	x_2	x_3	s_1	s_2
3	s_1=11/2	1/2	0	5/4	1	-1/4
0	x_2=17/2	1/2	1	3/4	0	1/4
	z_j	5/2	5	15/4	0	5/4
	z_j-c_j	-1/2	0	7/4	0	5/4

$\uparrow$

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_1 enters the basis

To find the variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if \ y_{ir} > 0\right\}$$

$$\theta_1 = \min\{(11/2)/(1/2), \quad (17/2)/(1/2)\} = \min\{11, 17\} = 11$$

[corresponding to the variable s_1]

$$\theta_2 = \min\left\{\frac{u_i - X_{Bi}}{y_{ir}}, if \ y_{ir} < 0\right\}$$

$$\theta_2 = \infty$$

$$\theta = \min\{\theta_1, \theta_2, u_1\} = \min\{11, \quad \infty, \quad 4\} = 4$$

$\theta = 4 = u_1$ [corresponding to the variable x_1]

Rule:-

If $\theta = u_r$ then variable x_r is not leaving the basis, but its upper bound value is calculated as $x_r = u_r - x_r'$, $0 \le x_r' \le u_r$

The new updated table is same as old one, but with the following two column changes

(a) x_r Column

Replace x_r by x_r', and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y=X_B$ column

$(x_{BK})_r = (x_{Bk})' - y_{kr} u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.]

Let $x_1=u_1-x_1'=4-x_1'$,   then

$0 \le 4 - x_1' \le 4$

$0 \le 4 - x_1' \le 4$

$0-4 \le 4 - x_1' - 4 \le 4-4$ (subtract 4)

$-4 \le -x_1' \le 0$

$4 \ge x_1' \ge 0$ (x by -1)

i.e $0 \le x_1' \le 4$

We know that

$x_{Bk}=x'_{Bk}-a_{kr}u_r$

$x_{B1}=x_{B1}'-y_{11}u_1=11/2-(1/2)(4)=7/2$

$x_{B2}=x_{B2}'-y_{21}u_1=17/2-(1/2)(4)=13/2$

$z_0=z_0'-(z_1-c_1)u_1=85/2-(-1/2)(4)=89/2$

	u_j	4	10	3	∞	∞
	c_j	-3	5	2	0	0
CB	X_B	x_1'	x_2	x_3	s_1	s_2
0	$s_1=7/2$	-1/2	0	5/4	1	-1/4
5	$x_2=13/2$	-1/2	1	3/4	0	1/4
	z_j	5/2	5	15/4	0	5/4
	z_j-c_j	1/2	0	7/4	0	5/4

Since all $z_j-c_j \ge 0$, therefore the above table is optimal

$x_1'=0$,

$4-x_1=0$ [since $x_1=4-x_1'$]

$x_1=4$,

Hence the optimal solution is given by

$x_1=4$, $x_2=13/2$, $x_3=0$ and Max $z=3(4)+5(13/2)+2(0)=89/2$.

Problem:-04

Use upper bound algorithm to solve

Max $z=4x_1+2x_2+6x_3$

Subject to

$4x_1-x_2\le 9$

$-x_1+x_2+2x_3\le 8$

$-3x_1+x_2+4x_3\le 12$

and $1\le x_1\le 3$, $0\le x_2\le 5$, $0\le x_3\le 2$

Solution:-

Since x_1 has *positive lower bound*

$1\le x_1\le 3$

$0\le x_1-1\le 3-1$ (Subtract 1)

$0\le x_1'\le 2$, where $x_1'=x_1-1$

Sub $x_1=x_1'+1$ in the given problem

The new updated problem is given by

Max $z=4x_1+2x_2+6x_3$

$=4(x_1'+1)+2x_2+6x_3$

$=4x_1'+2x_2+6x_3+4$

Subject to

$4(x_1'+1)-x_2\le 9$	$4x_1'+4-x_2\le 9$	$4x_1'-x_2\le 5$
$-(x_1'+1)+x_2+2x_3\le 8$	$-x_1'-1+x_2+2x_3\le 8$	$-x_1'+x_2+2x_3\le 9$
$-3(x_1'+1)+x_2+4x_3\le 12$	$-3x_1'-3+x_2+4x_3\le 12$	$-3x_1'+x_2+4x_3\le 15$
and $1\le x_1'+1\le 3$	$1-1\le x_1'+1-1\le 3-1$	$0\le x_1'\le 2$

$0\le x_2\le 5$, $0\le x_3\le 2$

Apply simplex algorithm

Max $z=4x_1'+2x_2+6x_3+4+0s_1+0s_2+0s_3$

Subject to

$4x_1'-x_2+s_1=5$ (add slack variable $s_1\geq 0$)

$-x_1'+x_2+2x_3+s_2=9$ (add slack variable $s_2\geq 0$)

$-3x_1'+x_2+4x_3+s_3=15$ (add slack variable $s_3\geq 0$)

and $0\leq x_1'\leq 2$ $0\leq x_2\leq 5$, $0\leq x_3\leq 2$, $0\leq s_1\leq \infty$, $0\leq s_2\leq \infty$ $0\leq s_3\leq \infty$.

The initial basic feasible solution is given by

$s_1=5$, $s_2=9$, $s_3=15$, $x_1'=0$, $x_2=0$, $x_3=0$ and Max $z=4(0)+2(0)+6(0)+4=4$

	u_j	2	5	2	∞	∞	∞
	c_j	4	2	6	0	0	0
CB	X_B	x_1'	x_2	x_3	s_1	s_2	s_3
0	$s_1=5$	4	-1	0	1	0	0
0	$s_2=9$	-1	1	2	0	1	0
0	$s_3=15$	-3	1	4	0	0	1
	z_j	0	0	0	0	0	0
	z_j-c_j	-4	-2	-6	0	0	0

Since some z_j-c_j are negative, therefore the above table is not optimal.

x_3 enters the basis

To find variable leaves the basis

$$\theta_1=\min\left\{\frac{X_{Bi}}{y_{ir}}, if\ y_{ir}>0\right\}$$

$$\theta_1=\min\{9/2, 15/4\}=15/4$$ [corresponding to the variable s_3]

$$\theta_2=\min\left\{\frac{u_i-X_{Bi}}{a_{ir}}, if\ a_{ir}<0\right\}$$

$\theta_2 = \infty$

$\theta = \min\{\theta_1, \theta_2, u_3\} = \min\{15/4, \quad \infty, \quad 2\} = 2$

$\theta = 2 = u_3$ [corresponding to the variable x_3]

Rule:-

If $\theta = u_r$ then variable x_r is not leaving the basis, but its upper bound value is calculated as $x_r = u_r - x_r'$, $0 \le x_r' \le u_r$

The new updated table is same as old one, but with the following two column changes

(a) x_r Column

Replace x_r by x_r', and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y = X_B$ column

$(x_{BK})_r = (x_{Bk})' - y_{kr} u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.]

Let $x_3 = u_3 - x_3' = 2 - x_3'$, then

$0 \le 2 - x_3' \le 2$,

$0 - 2 \le 2 - x_3' - 2 \le 2 - 2$ (subtract 2)

$-2 \le -x_3' \le 0$

$2 \ge x_3' \ge 0$ (x by -1)

i.e $0 \le x_3' \le 2$

We know that

$x_{Bk} = x'_{Bk} - a_{kr} u_r$

$x_{B1} = x'_{B1} - y_{31} u_3 = 5 - (0)(2) = 5$

$x_{B2} = x_{B2} - y_{32} u_3 = 9 - (2)(2) = 5$

$x_{B3} = x_{B3} - y_{33} u_3 = 15 - (4)(2) = 7$

$z_0 = z_0' - (z_3 - c_3) u_3 = 4 - (-6)(2) = 16.$

	u_j	2	5	2	∞	∞	∞
	c_j	4	2	-6	0	0	0
CB	X_B	x_1'	x_2	x_3'	s_1	s_2	s_3
0	s_1=5	[4]	-1	0	1	0	0
0	s_2=5	-1	1	-2	0	1	0
0	s_3=7	-3	1	-4	0	0	1
	z_j	0	0	0	0	0	0
	z_j-c_j	-4	-2	6	0	0	0

x_1' enters the basis

To find variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if\ \ y_{ir} > 0\right\}$$

$\theta_1 = \min\{5/4\} = 5/4$ [corresponding to the variable s_1]

$$\theta_2 = \min\left\{\frac{u_i - X_{Bi}}{y_{ir}}, if\ \ y_{ir} < 0\right\}$$

$$\theta_2 = \min\left\{\frac{u_1 - X_{B1}}{-y_{21}}, \frac{u_1 - X_{B1}}{-y_{31}}\right\} = \min\{\frac{2-5}{-(-1)}, \frac{2-7}{-(-3)}\} = \min\{-3, -5/3\} = \infty$$

$$\theta = \min\{\theta_1, \theta_2, u_1\} = \min\{5/4, \quad \infty, \quad 2\} = 5/4$$

$\theta = 5/4 = \theta_1$ [corresponding to the variable s_1]

s_1 leaves the basis

[Rule :-

If $\theta = \theta_1$ then corresponding basic variable is removed from the basis. Select the basic variable to leave the basis and apply usual simplex algorithm to get new updated table. *Go to Step 06.*]

First row values

5/4 4/4=1 -1/4 0/4=0 1/4 0/4=0 0/4=0

Second row values

5-(5)(-1)/4=25/4 1-(-1)(-1)/4=3/4 -2-(0)(-1)/4=-2

0-(1)(-1)/4=1/4 1-(0)(-1)/4=1 0-(0)(-1)/4=0

Third row values

7-(5)(-3)/4=43/4 1-(-1)(-3)/4=1/4 -4-(0)(-3)/4=-4

0-(1)(-3)/4=3/4 0-(0)(-3)/4=0 1-(0)(-3)/4=1

	u_j	2	5	2	∞	∞	∞	
	c_j	4	2	-6	0	0	0	
CB	X_B	x_1'	x_2	x_3'	s_1	s_2	s_3	
4	x_1'=5/4	1	[-1/4]	0	1/4	0	0	←
0	s_2=25/4	0	3/4	-2	1/4	1	0	
0	s_3=43/4	0	1/4	-4	3/4	0	1	
	z_j	4	-1	0	1	0	0	
	z_j-c_j	0	-3	6	1	0	0	
			↑					

x_2 enters the basis

To find variable leaves the basis

$$\theta_1 = \min\left\{\frac{X_{Bi}}{y_{ir}}, if \ \ y_{ir} > 0\right\}$$

$$\theta_1 = \min\{(25/4)/(3/4), (43/4)/(1/4)\} = \min\{25/3, 43\} = 25/3$$

[corresponding to the variable s_2]

$$\theta_2 = \min\left\{\frac{u_r - X_{Bi}}{-y_{ir}}, if \ y_{ir} < 0\right\}$$

$$\theta_2 = \min\left\{\frac{u_2 - X_{Bi}}{-y_{i2}}, if \ y_{i2} < 0\right\}$$

$$\theta_2 = \min\left\{\frac{5-5/4}{-(-1/4)}\right\} = \min\left\{\frac{15/4}{1/4}\right\} = 15$$ [corresponding to the variable x_1']

$$\theta = \min\{\theta_1, \theta_2, u_2\} = \min\{25/3, \quad 15, \quad 5\} = 5$$

$\theta = 5 = u_2$ [corresponding to the variable x_2]

Rule:-

If $\theta = u_r$ then variable x_r is not leaving the basis, but its upper bound value is calculated as $x_r=u_r-x_r'$, $0 \le x_r' \le u_r$

The new updated table is same as old one, but with the following two column changes

(a) x_r Column

Replace x_r by x_r', and multiply the x_r' column values by -1, because the coefficient of x_r' is negative in the substitution.

(b) $Y=X_B$ column

$(x_{BK})_r=(x_{Bk})'-y_{kr}u_r$,

Here $(x_{Bk})'$ is the old table value and (x_{Bk}) is the new table value.]

Let $x_2=u_2-x_2'=5-x_2'$, then

$0 \le 5-x_2' \le 5$,

$0-5 \le 5-x_2'-5 \le 5-5$ (subtract 5)

$-5 \le -x_2' \le 0$

$5 \ge x_2' \ge 0$ (x by -1)

i.e $0 \le x_2' \le 5$

We know that

$x_{Bk}=x'_{Bk}-a_{kr}u_r$

$x_{B1}=x'_{B1}-y_{12}u_2=5/4-(-1/4)(5)=5/2$

$x_{B2}=x_{B2}-y_{22}u_2=25/4-(3/4)(5)=10$

$x_{B3}=x_{B3}-y_{32}u_2=43/4-(1/4)(5)=19/2$

	u_j	2	5	2	∞	∞	∞
	c_j	4	-2	-6	0	0	0
CB	X_B	x_1'	x_2'	x_3'	s_1	s_2	s_3
4	$x_1'=5/2$	1	1/4	0	1/4	0	0
0	$s_2=10$	0	-3/4	-2	1/4	1	0
0	$s_3=19/2$	0	-1/4	-4	3/4	0	1
	z_j	4	1	0	1	0	0
	z_j-c_j	0	3	6	1	0	0

Since $z_j-c_j \geq 0$, therefore the above table is optimal.

Hence the optimal solution is $x_1'=5/2$, $x_2'=0$, $x_3'=0$

$x_1'=5/2$

$x_1-1=5/2$ [since $x_1'=x_1-1$]

$x_1=7/2$

$x_2'=0$

$5-x_2=0$ [since $x_2=5-x_2'$]

$x_2=5$

$x_3'=0$

$2-x_3=0$ [since $x_3=2-x_3'$]

$x_3=2$

i.e $x_1=7/2, x_2=5, x_3=2$ and Max $z=4(7/2)+2(5)+6(2)=36$

Exercise

Solve the following LP problems

1. Max $z=3x_1+2x_2$

Subject to constraints

$x_1-3x_2 \leq 3$, $x_1-2x_2 \leq 4$, $2x_1+x_2 \leq 20$, $x_1+3x_2 \leq 30$, $-x_1+x_2 \leq 6$

and $0 \leq x_1 \leq 8$; $0 \leq x_2 \leq 6$. **Ans. $x_1=7$, $x_2=6$, max $z=33$.**

2. Max $z=3x_1+5x_2+2x_3$

Subject to constraints

$x_1+2x_2+2x_3 \leq 14$, $2x_1+4x_2+3x_3 \leq 23$

and $0 \leq x_1 \leq 4$; $2 \leq x_2 \leq 5$, $0 \leq x_3 \leq 3$. **Ans. $x_1=0$, $x_2=15/4$, max $z=123/4$.**

3. Max $z=2x_1+x_2$

Subject to constraints

$x_1+2x_2 \leq 10$, $x_1+x_2 \leq 6$, $x_1-x_2 \leq 2$, $x_1-2x_2 \leq 1$

and $0 \leq x_1 \leq 3$; $0 \leq x_2 \leq 2$. **Ans. $x_1=3$, $x_2=2$, max $z=8$.**

4. Max $z=4x_1+4x_2+3x_3$

Subject to constraints

$-x_1+2x_2+3x_3 \leq 15$, $-x_2+x_3 \leq 4$, $2x_1+x_2-x_3 \leq 6$, $x_1-x_2+2x_3 \leq 10$

and $0 \le x_1 \le 8$; $0 \le x_2 \le 4$, $0 \le x_3 \le 4$. **Ans. $x_1=17/5$, $x_2=16/5$, $x_3=4$ max $z=192/5$.**

5. Min $z=x_1+2x_2+3x_3-x_4$

Subject to constraints

$x_1-x_2+x_3-2x_4 \le 6$, $-x_1+x_2-x_3+x_4 \le 8$, $2x_1+x_2-x_3 \ge 2$

and $0 \le x_1 \le 3$; $1 \le x_2 \le 4$, $0 \le x_3 \le 10$, $2 \le x_4 \le 5$

Ans. $x_1=17/5$, $x_2=16/5$, $x_3=4$ max $z=192/5$.

UNIT-III

Inventory control

The word inventory refers to any kind of resources having economic value and is maintained to full fill the present and further needs of the organisation.

Fred hansman inventory as an ideal resources of any kind provided such a resource has economic.

Resources

Resources may be classified into three categories.

(i) physical resources such as raw material, semi-finish goods, finished goods,

Spare part, lubricant etc...

(ii) Human resources such as un-used labour

(iii) Financial resource such as working capital etc..

Example

Types of organisation	**Type of inventory held**
Manufacture	Raw material, spare parts,semi-finished goods,finished good etc...
Hospital	No. Of beds, stock of drugs,specialised doctor etc...
Bank	Cash reserve etc..
Air line company	Seating capacity spare parts etc..

Use of inventory

It is essential to balance the advantage of having inventory of resources and the cost of maintain it so as to determine an optimal level of inventory of each resources so that the total inventory cost is minimum.

Inventory models

The model of inventory are classified into two catogeries.

(i) deterministic model

(ii) probability model

Deterministic model

Deterministic model deals with constant rate of demand and lead time situation to decided ordering quantity (EOQ) so that total cost of ordering (c_p),

Inventory holding cost (c_h) and shortage cost(c_s) is at its minimum.

Probability model

Probabilistic model deals with though situation to determine EOQ in which demand/lead time is probabilistic .

Relevant cost

The cost that are affected (i.e, increase or decrease) by the company decision to maintain particular level of inventory are called relevant cost.

Purchase cost

This cost consists of the actual price paid for the procurement cost

(i) when the unit price (c)of an item is independent of the size of the quantity order or purchased.

Purchased cost = (price/unit)*(demand/unit time)

C =C.D

(ii) when price break or quantity discounts are available for bulk purchase above a specified quantity, the unit price become smaller as size of order quantity exceeds the specified level.

In this case the purchase cost become variable and depends on the size of the order.

Purchase cost=(price/unit when order size is q)* (demand per unit time)

$$=c(Q).D$$

Carrying or holding cost

The carrying cost is associated with carrying or holding inventory

(i) carrying cost=(cost of carrying 1 units of an item in the inventory for a given length of time) * (average number of units of an item carrying in the inventory for a given length of time).

(ii) carrying cost = (cost to carry one rupee worth of inventory per time period)* (rupees value of unit carried)

Note

carrying cost includes

(i) storage cost for providing ware house.

(ii) inventory holding cost for payment of salaries to employees.

(iii) Insurance cost against possible loss from fire or other form of damage etc.

Ordering cost(setup cost)

Ordering cost incurred each time an order is placed for procuring items from outside suppliers.

When an item is produced internally ordering cost is reffered as setup cost which includes both paper cost and physical preparation cost.

Ordering cost =(cost per order)*(number of orders in the inventory planning period).

Setup cost= (cost per setup)*(no of setup in the planning periods)

Note

Ordering cost includes

(i) requisition cost of handling invoice, stationary, papers, etc..

(ii) cost of services like mailing, telephone calls etc..

Shortage or stock out cost

Shortage of items occur when item cannot be supplied o demand.

(i) the supply of items is awaited by the customers

i.e, the items are back ordered and therefore there is no loss in sale.

In this case it is very difficult to determine the nature and magnitude of the back ordering cost.

The situation may lead to loss of good will.

(ii) customers are not ready to will therefore there the loss of sale.

In this case shortage cost shall be measured in terms of good eill loss

Shortage cost=(cost of being short one unit in the inventory)*(average no of units short in the inventory)

Average number if unit short= $\frac{(\min inventory)+(\max inventory)}{3}*$ (time for which shortage occurs)

Total inventory cost= purchase cost + ordering cost + carrying cost+ shortage cost

When the purchase cost remains constant and is independent of the quantity of purchase.

Total variable inventory cost(TVC) = ordering cost + carrying cost + shortage cost

1) optimal length if inventory replenishment cycle time (t^*), optimal interval between the successive orders.

Q^* = annual demand * reorder cycle time

=D.t

Or

$$t^* = \frac{Q^*}{D} = \frac{1}{D} * \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2C_0}{DC_h}}$$

2) optimal number of orders (N^*) to the placed in the given time period (assumed as one year)

$$N^* = \frac{D}{Q^*} = D * \frac{1}{\sqrt{\frac{2DC_0}{C_1}}} = \sqrt{\frac{DC_h}{2C_0}}$$

3) optimum (minimum) total variable inventory cost(TVC^*)

$$TVC^* = \frac{D}{Q^*} C_0 + \frac{Q^*}{2} C_h = D.C_0 * \frac{1}{\sqrt{\frac{2DC_0}{C_h}}} + \frac{C_h}{2} * \sqrt{\frac{2DC_0}{C_h}}$$

$$= \sqrt{2DC_0 C_h}$$

4) optimal total inventory cost is the seem of variable cost and fixed cost.

$TC = DC + TVC^*$

Alternative formulas:

TVC= 2* reorder cost component

=2*holding cost component

$$TVC = \frac{2DC_0}{Q^*} = Q^* * C_h$$

Carrying cost = inventory carrying rate * unit cost of item

= r * C

Demand in units= rupee value of demand / unit cost of the item

If unit cost of item is not known then EOQ in rupee terms is expressed as

$$Q^* = \sqrt{\frac{2 * annualdema\ ndinrupee}{inventoryc\ arrying\ \cos t}}$$

$$= \sqrt{\frac{2 * C.D * C_0}{r * C}}$$

Lead time

Elapsed time between the placement of order and its receipt in the inventory is known as lead time.

Reorder time

This is the time when we should place an order by taking into consideration the interval between placing the order and receiving the supply.

Example

We like to place a new order precisely at the time when the inventory level reaches zero.

Economic order quantity(EOQ)

Economic order quantity is the size of order which minimize the total inventory cost of carrying inventory and the ordering cost.

Under the assumed conditions of certainty and the annual demand are known.

Problem

The production department for a company requires 3600 kg of raw material for manufacturing a particular item for a year it has been estimated that the cost of placing an order is rs36 and cost of carrying inventory is 25% of the investment in the inventory. The price is rs 10/kg. The purchase manager wishes to determine an ordering policy for raw material.

Solution

D=3600 kg/year

C_0 = Rs36 /order

C_h=25% of investment

C=10

C_h=r*C

$$\frac{25}{100}*10$$

=Rs.2.5 /kg/year

$$Q^* = \sqrt{\frac{2*D*C_0}{C_h}}$$

$$= \sqrt{\frac{2*3600*36}{2.5}}$$

=321.99 kg/order

$$t^* = \sqrt{\frac{2C_0}{DC_h}}$$

$$\sqrt{\frac{2*36}{3600*2.5}}$$

=0.089 / year

$$N^* = \sqrt{\frac{DC_h}{2C_0}}$$

$$= \sqrt{\frac{3600*2.5}{2*36}}$$

=11.18 order/year

$TVC^* = \sqrt{2 D C_0 C_h} = \sqrt{2 * 3600 * 36 * 2.5}$

=804.98 / year

$TC = DC + TVC^*$

=(3600*10)+804.98

=36804.98 / year

Formula

Profit=revenue-total cost

Revenue= demand*selling price

2) A company operating 50 weeks in a year concened about its stock of copper cable. This cost rupees 240/meter and there is a demand for a 8000m a week. Each replenishment cost Rs1050 for administer and Rs1650 for delivery holding cost are estimated at 25% of the value held a year. Assuming no shortage are allowed. What is the optimal inventory policy for the company. How would this analysis f=differ if the company wanted to maximize the profit rather than minimize cost? What is the gross profit if the company sell cable for Rs.360 a meter.

Solution

C=purchase cost = 240/meter

D= 8000*50=no of working weeks* demand/week

=400000 m/year

C_0= Administer cost +delivering cost

=1050+1650 =Rs 2700m/year

C_h=r*C

$= \frac{25}{100} * 240$

=Rs. 60m/year

Q*= $\sqrt{\frac{2*D*C_0}{C_h}}$

= $\sqrt{\frac{2*400000\ *2700}{60}}$

=6000 m/year

$t^* = \sqrt{\frac{2C_0}{DC_h}}$

$\sqrt{\frac{2*2700}{400000\ *60}}$

=0.015/ year

TVC*= $\sqrt{2DC_0C_h}$ = $\sqrt{2 * 400000 * 2700 * 60}$

=360000 / year

TC=DC+TVC*

=(400000*240)+360000

=96360000m / year

If the company sell the cable for a Rs.360m its revenue is Rs360*400000 (selling cost* demand)

=Rs14400000m/year

Profit= revenue-total cost

= 144000000-96360000

=47640000

3) An aircraft company uses rivets at an approximately constant rate of 5000kg/y. The rivets cost rupees 2/kg and the company personal estimate that it cost Rs 200 to place an order and the sharing cost of inventory is 10%/y

(i) How frequently should order for rivets be placed and what quantity should be ordered for

(ii) If the actual cost for Rs 500 to place an order and 15% for carrying cost, the optimal policy would change how much the company losing per year because of imperfect cost of information.

Solution

D=5000kg/y

C=20/kg

r=inventory carrying rate =10%=0.1

C_0=rs200/order

C_h=r*c=0.1*20=rs2/year

(i)

$$Q^* = \sqrt{\frac{2 * D * C_0}{C_h}}$$

$$= \sqrt{\frac{2 * 5000 * 500}{2}}$$

=1000/order

$N^* = \frac{D}{Q^*}$ $\frac{5000}{1000}$

=5

TVC*= $\sqrt{2DC_0C_h} = \sqrt{2*5000*200*2}$

=2000/ year

TC=DC+TVC*

=(500*20)+2000

=10200 / year

(ii)

If r =15%=0.15

C_0=rs500/order

C_h=r*c=0.15*20=3

$Q^* = \sqrt{\frac{2*D*C_0}{C_h}}$

$= \sqrt{\frac{2*5000*500}{3}}$

=1291/order

TVC*= $\sqrt{2DC_0C_h} = \sqrt{2*5000*500*3}$

=3873/ year

Thus the loss per year due to imperfect cost information

= 3873-2000

=1873/yr

Note

(i) TC =10200

(ii) TC=D.C+TVC

=(5000*20)+3873

=103873

=103873-10200

=1873/yr

4) Retail store sells 5200 units of production in a year. Each unit cost Rs.2 to this store. The whole saler charges Rs.10 for each delivery irrespective of the quantity order. The interest charges on working capital 15% and the insurance charges on inventory amount to 5% per annum. All other expenses either fixed in nature pr do not vary with the level of inventory or the quantity order. The owner is presently following the policy of ordering 100 units every week.

He wish to evaluate this inventory policy. What recommendation would you make?

solution

D=5200units/year

C=2

C_0 = Rs10 /order

Holding cost =15%+5%

r=20%

C_h=r*C=0.4

$Q^* = \sqrt{\frac{2 * D * C_0}{C_h}}$

$= \sqrt{\frac{2 * 5200 * 10}{0.4}}$

=510 units

$t^* = \sqrt{\frac{2C_0}{DC_h}}$

$\sqrt{\frac{2 * 10}{5200 * 0.4}}$

=0.098/ year

$N^* =$ D/Q^*

$= \frac{5200}{510} = 10$

Note

$TVC^* = \sqrt{2DC_0C_h} = \sqrt{2 * 5200 * 10 * 0.4}$

=203.9

=204/yr

$TC = DC + TVC^*$

=(5200*2)+204

=10604 / year

5) A company works 50 weeks in a year. For a certain part included in the assumble of several part. There is a annual demand of 10000 units. this may be obtained from either an outside supplier or subsidiary company. The following data relating to the parts are given

	From the outside suppliers Rs.	From subside company
Purchase price/unit	12	13
Cost of placing an order	10	10
Cost of receiving order	20	15
Storage an all carrying cost included capital cost /unit/annum	2	2

(i) What purchase quantity and from which source would you recommend

(ii) What would be the minimum total cost.

Solution

D=10000 units

From outside supplies

C=12/unit

C_0 = 10+20=30/order

C_h=2

$Q^* = \sqrt{\dfrac{2 * D * C_0}{C_h}}$

$= \sqrt{\dfrac{2 * 10000 * 30}{2}}$

=548unit

$TVC^* = \sqrt{2DC_0C_h} = \sqrt{2*10000*30*2}$

$=1095$/unit

$TC=DC+TVC^*$

$=(10000*12)+1095$

$=121095$/unit

From subside company

D=10000 units

C=13/unit

$C_0 = 10+15=25$/order

$C_h=2$

$$Q^* = \sqrt{\frac{2*D*C_0}{C_h}}$$

$$= \sqrt{\frac{2*10000*25}{2}}$$

=500unit

$TVC^* = \sqrt{2DC_0C_h} = \sqrt{2*10000*25*2}$

$=1000$/unit

$TC=DC+TVC^*$

$=(10000*13)+1000$

=131000/unit

Since the total cost of bying from subsiding company is more than the outside supplier.

Therefore an order of Q^*= 548 units should be place in outside supplies.

6) The whole seller supply 30 stuffed dolls each week days to varies shops. dolls are purchased from the manufacture in lots of 120 each per Rs.1200per lot. Every order incures a handling charge of Rs.60 plus the flight charge of Rs 25-per lot multiple and fractional lots also can be order and incremental cost is Rs0.60/yr to store the doll in the inventory. The whole seller finances inventory investment by paying its holding its company 2% monthly for borroded funds how many doll should be ordered for at a time in order to minimum the total annual inventory cost? Assume that there are 250 working days in a year how frequently should be ordered.

Solution

D=30/day

D=30*250=7500 dolls/year

Since manufacture pay rupees 1200 lot

C= cost/doll=Rs10 / doll

C_0=60+250=Rs310 / order

R=2% /month=2*12=24% /yr

C_h=rC=0.60+rC

=Rs 3/year

$$Q^* = \sqrt{\frac{2 * D * C_0}{C_h}}$$

$$= \sqrt{\frac{2 * 7500 * 310}{3}}$$

=1245dolls/order

$$t^* = \sqrt{\frac{2C_0}{DC_h}}$$

$$\sqrt{\frac{2 * 310}{7500 * 3}}$$

=0.166/ year

Note

TVC= $\frac{D}{Q^*} C_0 + \frac{Q^*}{2} C_h$

TC=D*C+TVC

TVC= annual ordering cost+annual carrying cost

7) A chemical company hold its inventory of raw material in special containers, it each container occupying 10sq feet of floor space. There are only 5000sq.feet of storage space available. Each year this company uses 9000 special containers of raw material paying Rs 8/ container of raw material. If ordering cost is Rs40/order and annual holding cost are 20% of the average inventory value, how much is it work for this company to increase its container of raw material storage of inventory can be stored with 5000sq.f storage limitation, assume that this company works a 300days /yr.

Solution

D=9000 containers/year

C=Rs 8/ containers

C_0 = 40 /order

R=20%=0.2

C_h=r*C=0.2*8=1.6

$$Q^* = \sqrt{\frac{2*D*C_0}{C_h}}$$

$$= \sqrt{\frac{2*9000*40}{1.6}}$$

=671 containers/order

Since each containers occupies 10sq.ft of floor space.

Therefore the company requires 671*10=6710 sqft of floor space

The total storage space available with the company is 5000sqft.

To calculate how much is work this company calculate TVC for Q^* =500 and Q^* =671.

$$\text{TVC} = \frac{D}{Q^*}C_0 + \frac{Q^*}{2}C_h$$

$$= \frac{9000}{500}40 + \frac{500}{2}(1.6)$$

=720+400

=1120

$$\text{TVC} = \frac{9000}{671}40 + \frac{671}{2}(1.6)$$

536.5+536.8

=1074

The company will save Rs 46 if it following economic order policy (Q^*=671).

Rs46 worth for the company to increase the storage space.

With 5000sqft limitation only 500 units can be order.

Each day usage is 30

Nearly 500/30=17 days supply of inventory 17 days of inventory will be there.

Model –III

EOQ model with different rate of demand

Annual carrying cost = $\frac{1}{2} q C_h T$

Annual ordering cost = $\frac{D}{q} C_0$

TVC= A.C.C+ A.D.C

$$= \frac{1}{2} q C_h T + \frac{D}{q} C_0$$

EOQ $q^* = \sqrt{\frac{2DC_0}{TC_h}}$

$TVC^* = \sqrt{2C_h C_0 (\frac{D}{T})}$

Where T is the total time period.

D is the total demand over the time T.

Model-IV

Economic production quantity model

When supply is gradual.

Formula

Annual carrying cost = $\frac{Q}{2}\left(1-\frac{d}{p}\right)C_h$

Production setup cost = $\frac{D}{Q}C_0$

Economic batch quantity (EBQ)

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)}$$

TVC*=carrying cost + production setup cost

$$= \frac{Q}{2}\left(1-\frac{d}{p}\right)C_h + \frac{D}{Q}C_0$$

$$= \sqrt{2DC_0C_h\left(1-\frac{d}{p}\right)}$$

Optimum length of each lot size production run

$$t_p^* = \frac{Q^*}{p}$$

Optimal production cycle time

$$t = \frac{Q^*}{D}$$

optimal number of production cycles.

$$N^* = \frac{D}{Q^*}$$

Note

p-production rate per day

d-demand rate per day

D-demand per day

Problem

1) A contractor has to supply 10000 bearings per day to an automobile manufacture. If finds that when his state production run, he can produce 25000 bearings per day. The cost of bearing in stock for a year Rs2 and setup cost of a production run is a rs1800. How frequently should production run we made.

Solution

d= 10000/day

p=25000/day

C_h=2

C_0=1800/production run

D= demand per day* no of working days

=10000*300=3000000

(assuming no of working day 300)

Economic batch quantity for each production

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)} \quad (p>d)$$

$$= \sqrt{\frac{2(3000000)1800}{2}\left(\frac{25000}{25000-10000}\right)}$$

=94868 bearings

$t^* = \frac{Q^*}{D}$ frequency of production cycle.

$$= \frac{94868}{25000}$$

=9.48 days

2) The manufacturing company use an EOQ approach in planning its production of gears. The following information is available. Each gear cost Rs 250 per year, annual demand is 60000 gears, setup cost are Rs 4000 per setup, and the inventory carrying cost per month is established at 2% of the average inventory value. When in production this gears can be produced at the rate of 400 units per day and these company works only for 300days in year. Determine economic lot size, the no of production run per year and the total inventory cost.

Solution

Since each gear cost Rs250

C=Rs250

D=annual demand=60000

d= D/no.of working days=600000/300

=200 gears/day

C_0=4000/setup

Carrying cost per month is establish at 2% of carrying inventory value.

Carrying cost per year = 2*12=24%

C_h=r x C=$\frac{24}{100} * 250$=60 per year

p= Rs400/day

Economic production per each lot size

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)}$$

$$= \sqrt{\frac{2(60000)4000}{60}\left(\frac{400}{400-200}\right)}$$

=4000years

No.of production run $N^* = \frac{D}{Q^*}$

$$N^* = \frac{60000}{4000} = 15$$

$$\text{TVC} = \frac{Q^*}{2}\left(1 - \frac{d}{p}\right)C_h + \frac{D}{Q}C_0$$

$$= \frac{4000}{2}\left(1 - \frac{200}{400}\right)60 + \frac{60000}{4000}(4000)$$

=Rs120000/yr

TC=TVC

= Rs120000/yr

3) The product is sold at the rate of 50 piece per day and is manufactured at the rate of 250 pieces per day. The setup cost for the machine is rs1000 and the storage cost is found to be rs.0.0015 per piece per day. With lower charges of Rs3.20 per piece and material cost Rs. 2.10 per piece overhead cost of Rs4.10 per piece, find minimum cost batch size with interest charges are 8%(assume 300 working days in a year). Compute the optimum numbers of cycle required. In a year for the manufacture of this product.

Solution

d= 50 piece per day

D=d*300=50*300=15000piece per year

p=250 piece per day

C_0=1000

storage cost is found to be rs.0.0015 per piece per day

storage cost 0.0015x300 per year

With lower charges of Rs3.20 per piece and material cost Rs. 2.10 per piece and overhead cost of Rs4.10 per piece .

i.e C=3.20+2.10+4.10=9.40

r=8%

C_h=r.C+0.0015(300)

C_h=(8/100).(9.4)+0.0015(300)

=1.202per year

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)}$$

$$= \sqrt{\frac{2(15000)1000}{1.202}\left(\frac{250}{250-50}\right)}$$

=5586

$$N^* = \frac{D}{Q^*} = \frac{15000}{5586}$$

=3cycles

4)a) At present a company is purchase at item X from outside suppliers. The consumption is 10000unit per year. The cost of the item is Rs5 per unit and the ordering cost is estimated to be Rs 100 per order. The cost carry inventory is 25%. If the consumption rate is uniform determine the economic purchase quantity.

b) In the above problem assume that company is going to manufacture the item with the equipment that is estimated to produce 100 units per day. The cost of the unit thus produced is Rs.3.5 per unit. A setup cost is Rs150 per setup and the inventory carrying charge is 25%. How has your answer changed.

Solution

(a)

D= 10000/year

C=Rs. 5 /unit

C_0=100/order

r=25%=0.25

C_h=rC=0.25*5=1.05

$$Q^* = \sqrt{\frac{2DC_0}{C_h}}$$

$$= \sqrt{\frac{2*100000*100}{1.05}}$$

=1265 unit per order

b)

D=10000/year

p=100 units per day

C=3.5 per unit

C_0=150 per setup

r=25%=0.25

C_h=rC=0.25*3.5=0.875%

d=10000/250=40unit per day

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)}$$

$$Q^* = \sqrt{\frac{2*10000*150}{0.875}\left(\frac{100}{100-40}\right)}$$

=2391 units per order

The increase in EOQ may be due to increase procurement cost(setup cost)

Model-V

EOQ model with constant rate of demand and variable order cycle time.

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_s}{C_s}\right)} \quad \text{(EOQ)}$$

$$M^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_s}{C_h + C_s}\right)} \quad \text{(optimal level)}$$

$$R^* = Q^* - M^* = Q^* \left\{\frac{C_h}{C_h + C_s}\right\} \quad \text{(optimum shortage level)}$$

$$t = \frac{Q^*}{D} = \sqrt{\frac{2DC_0}{DC_h}\left(\frac{C_h + C_s}{C_s}\right)}$$

TVC=order cost + carry cost+ shortage cost

$$= \frac{D}{Q}C_0 + \frac{M^2}{2Q}C_h + \frac{(Q-M)^2}{2Q}C_s$$

TC=D.C+TVC

$$TVC^* = \sqrt{2DC_0C_h\left(\frac{C_s}{C_s + C_h}\right)}$$

Problem

1)consider the following data

Unit cost	Rs 100
Order cost	Rs 160
Inventory carry cost	Rs 20
Back order cost(due to stock out)	Rs 10
Annual demand	1000units

i) minimum cost order quantity

ii)time b/w order

iii)maximum no. Of back order

iv) maximum level of inventory

v)overall annual cost

Solution

$C=100$

$C_0=160$

$C_h=20$

$C_s=10$

$D=1000$

(i)

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_s}{C_s}\right)}$$

$$= \sqrt{\frac{2*1000*160}{20}\left(\frac{20+10}{10}\right)}$$

=219

(ii)

$$t = \frac{Q^*}{D}$$

$$= \frac{219}{1000}$$

=0.219 yr (0.219*12=2.6months)

(iii) maximum back order = maximum no.of shortage

$$R^*=Q^* \left\{\frac{C_h}{C_h + C_s}\right\}$$

=146units

(iv) maximum no.of inventory level=optimal stock level

$$M^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_s}{C_h + C_s}\right)}$$

$$= \sqrt{\frac{2*1000*160}{20}\left(\frac{10}{20+10}\right)}$$

=73

(v) TC=D.C+TVC

$$\text{TVC}^{*}= \sqrt{2DC_0C_h\left(\frac{C_s}{C_s + C_h}\right)}$$

$$\text{TVC}^{*}= \sqrt{2*1000*160*20\left(\frac{10}{10+20}\right)}$$

=1460

TC= (1000*100)+1460

=101460

2)A commodity is to be supply at a constant rate of 200 units per day. Supply of any amount can be obtained at any required time, but each ordering cost Rs50

Cost of holding the commodity in inventory is Rs 2 per day while the delay in supply of item induce the penalty of Rs 10 per unit per day. Find the optimum policy if the penalty cost become infinity.

Solution

d=200/day

C_0=Rs50/order

C_h=Rs2unit/day

C_s=Rs10 unit/day

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_s}{C_s}\right)}$$

$$= \sqrt{\frac{2*200*50}{2}\left(\frac{12}{10}\right)}$$

=109.5 units

$$t = \frac{Q^*}{D}$$

$$= \frac{109.5}{200} = 0.547$$

If $C_s = \infty$ then

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_s}{C_s}\right)}$$

$$= \sqrt{\frac{2*200*50}{2}\left(\frac{C_h + C_s}{C_s}\right)}$$

$$= \sqrt{\frac{2*200*50}{2}\left(\frac{C_h}{C_s} + 1\right)}$$

$$= \sqrt{10000\left(\frac{2}{\infty} + 1\right)}$$

$$= \sqrt{10000(1)}$$

=100 units

3) The dealer supply you the following information with regard a product deal in by you. Annual demand 10,000units ordering cost Rs 10/order, price

Rs 20/unit, inventory carrying cost 20% of the value of inventory per year. The dealer is considering the possibilities of allowing some back order (stock out) to occur. He has estimated that annual cost of back ordering will be 25% of value of inventory.

(i) what should be the optimum no.of units of the product the should by in one lot?

(ii) What quantity of the product should be allowed to be back order, if any?

(iii) What could be the max quantity of inventory at the any time of the year?

(iv) would be requirement to allow back ordering?

If so, what would be the annual cost saving by adopting the policy of back ordering

Solution

D=10,000/year

C_0=Rs10/order

C=Rs20/unit

r=20%

C_h=r.C=0.20*20=4 per unit per year

C_s=25% of Rs. 20=Rs5 per unit per year [C=20]

(i) economic back order quantity

When stock out not permitted

$$Q^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2*10000*10}{4}}$$

=223.6 units

When back order is permitted

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_s}{C_s}\right)}$$

$$Q^* = \sqrt{\frac{2*100000*10}{4}\left(\frac{9}{5}\right)}$$

=300 units

(ii) optimum quantity of product to be back order (R^*)

$R^*=Q^* \left\{\frac{C_h}{C_h + C_s}\right\}$

$=300* \left(\frac{4}{9}\right)$

=133.3 unit/day

(iii) max quantity of inventory at the any time of the year(M^*)

$$M^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_s}{C_h + C_s}\right)}$$

$$= \sqrt{\frac{2*1000*10}{4}\left(\frac{5}{9}\right)}$$

=167 units/day

(iv)

TVC=

$\sqrt{2DC_0C_h}$

$= \sqrt{2*10000*10*4}$

=894.4 units/day (Q^*=223.6)

$$TVC(Q^* = 300) = \sqrt{2DC_0C_h\left(\frac{C_s}{C_h + C_s}\right)}$$

=666.6 units/day

TVC(Q^*=223.6)>TVC(Q^*=300)

Therefore the dealer should accept the proposal for back ordering and this will save him.

Rs 227.76/yr (894.4-666.6)

4) A dealer suppliers you the following information will regard a product deal in by you annual demand 5000 units buying cost Rs 250 per order, C_h is 30% per year, C =Rs100/ unit. The dealer is consider the possibilities of allowing some back order(stock out) to occur for the product. The estimated that annual cost of back ordering (allowing shortage) of the product will be Rs10/ unit.

(i) what should be the optimum no.of units of the product he should by in one lot?

(ii) what quantity of the product should be allowed to be back order?

(iii) how much additional cost will be have inquire on inventory if he does not permit back ordering?

Solution

D=5000 units

C_0=Rs250/order

C_h=r.C=100*0.30=30 /unit/year

C_s=Rs10 unit

(i)

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_s}{C_s}\right)}$$

$$= \sqrt{\frac{2*250*5000}{30}\left(\frac{30+10}{10}\right)}$$

=576 units

(ii)optimum level of inventory

$$M^* = Q^* \left\{\frac{C_s}{C_s + C_h}\right\}$$

=576{10/40}

=144 units

The quality of product to back order $R^* = Q^* - M^*$

=576-144

=432 units

(iii) Additional cost incurred=TVC=

$$= \sqrt{2DC_0C_h\left(\frac{C_s}{C_h + C_s}\right)}$$

$$= \sqrt{2*5000*250*30\left(\frac{10}{40}\right)}$$

=Rs 4330

EOQ model with gradual supply and shortage allow

Optimal production lot size

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)\left(\frac{C_h + C_s}{C_s}\right)}$$

Optimal level of shortage

$$R^* = Q_2^* = Q^* \left(1 - \frac{d}{p}\right)\left(\frac{C_h}{C_s + C_h}\right)$$

Production cycle time

$$t = \frac{Q^*}{D}$$

$$t^* = \sqrt{\frac{2C_0}{DC_h}\left(\frac{p}{p-d}\right)\left(\frac{C_h + C_s}{C_s}\right)}$$

Optimal inventory level

$$Q_1^* = \left(\frac{p-d}{p}\right)Q^* - Q_2^*$$

$$Q_1^* = \sqrt{\frac{2DC_0}{C_h}\left(1 - \frac{p}{d}\right)\left(\frac{C_s}{C_s + C_h}\right)}$$

Total minimum variable inventory cost

$$TVC^* = \sqrt{2DC_0C_h\left(1 - \frac{d}{p}\right)\left(\frac{C_s}{C_s + C_h}\right)}$$

Problems

1) The cost parameter and other factors for a production inventory system of automobile pitons of given below demand per year D=6000 units, unit per cost C=Rs40, setup cost C_0=Rs500, C_h=Rs8, production rate per year 36000units shortage cost per unit per year C_s=Rs20

Find (i) optimum lot size

(ii) no.of shortages

(iii) manufacturing time

(iv) time between setups cycle time.

Solution

D=6000 units /yr

P=36000/yr

C=Rs40

C_0=Rs500

C_h=Rs8

C_s=Rs20

(i)

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)\left(\frac{C_h+C_s}{C_s}\right)}$$

$$Q^* = \sqrt{\frac{2*6000*500}{8}\left(\frac{36000}{36000-6000}\right)\left(\frac{8+20}{20}\right)}$$

=1123 units

(ii)

$$R^{*}=Q^{*}\left(1-\frac{d}{p}\right)\left(\frac{C_h}{C_s+C_h}\right)$$

$$=1123\left(1-\frac{6000}{36000}\right)\left(\frac{8}{8+20}\right)$$

=267 units

(iii)

$$t^{*}=\frac{Q^{*}}{D}$$

$$=\frac{1123}{6000}$$

=0.19yrs

$$t^{*}=\frac{Q^{*}}{P}$$

$$=\frac{1123}{36000}$$

=0.03yrs

problem:-02

The demand for an item in a company is 18000 units per year, and the company can produce the at the rate of 3000 per month. The cost of one setup is Rs 500 and the holding cost of one unit per month is 15 piece the shortage cost of one unit is Rs240 per year. Determine the optimum manufacturing quantity and the no.of shortage. Also determine the manufacturing time and the time b/w setup.

Solution

D=18000 units /yr

d=1500 unit per month

p=3000/month

C_0=Rs500/setup

C_h=0.15 paisa per unit per month

C_s=20 per month

(i)

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)\left(\frac{C_h + C_s}{C_s}\right)}$$

$$Q^* = \sqrt{\frac{2*1500*500}{0.15}\left(\frac{3000}{3000-1500}\right)\left(\frac{0.15+20}{20}\right)}$$

=4489 per month

(ii)

$$R^* = Q^{*\left(1-\frac{d}{p}\right)\left(\frac{C_h}{C_s + C_h}\right)}$$

$$= \quad 4489^{*\left(1-\frac{1500}{3000}\right)\left(\frac{0.15}{0.15+20}\right)}$$

=17 per month

(iii)

$$t^* = \frac{Q^*}{d}$$

$$t^* = \frac{4489}{1500}$$

=3 month(time b/w setup)

(iv)

$$t^* = \frac{Q^*}{P}$$

$$t^* = \frac{4489}{3000}$$

=1.5 month(manufacturing time)

3) A purchase as desired to place order for a quantity of 500 units of a particular item in order to get discount of 10%. From the past record it was found out that in the last year 8 orders each of size 200 units where placed. Given the ordering cost Rs 500/order inventory cost 40% of the inventory value and the price of the item of Rs400/unit. Is the purchase manager justified in the decision? What is the exact of his decision of the company.

Solution

D=8*200=1600 units /yr

C_0=Rs500/order

r=40%

C=400/unit

C_h=r.C=.40x400=160

$$Q^* = \sqrt{\frac{2DC_0}{C_h}}$$

$$= \sqrt{\frac{2*1600*500}{160}}$$

=100units

Price break problem

1)A shop keeper has a uniform demand of an item at the rate of 100 item per month. He buys from supplier at a cost of Rs12 per item and the cost of ordering is Rs 10 each time. If the stock holding cost are 20% per year of stock value. How frequently should he replenish his stock? Further suppose the suppliers offers 5% discount on orders between b/w 200 and 999 item,10% discount on orders exceeding or equal to 1000, can the shopkeeper reduces his cost by taking advantage of either of those discount.

Solution

d=100/month

D=1200/yr

C=12/item

C_0=Rs. 10/yr

r=20%=0.20

C_h=rC=0.20*12=2.4

When no discount is offer

$$Q^* = \sqrt{\frac{2DC_0}{C_h}}$$

$$= \sqrt{\frac{2*1200*10}{2.4}}$$

=100

$$N^* = \frac{D}{Q^*}$$

$$= \frac{1200}{100}$$

=12

$$TVC = \sqrt{2DC_0C_h}$$

$$= \sqrt{2*1200*10*2.4}$$

=240

TC=D.C+TVC

=(1200*12)+240

=14640

(ii)

When quantity discount are offer

Quantity	prince/unit
$0 \le Q_1 < 200$	12 (no discount)
	5% of 12=0.05x12=0.6
$200 \le Q_2 < 1000$	11.40 (5% discount 0.6rs)
	10% of 12 =.1x12=1.2
$1000 \le Q_3$	10.8 (10% discount 1.2rs)

b_1=200

b_2=1000

let us calculate Q_3^* based on the price C_3=10.8

$$Q^* = \sqrt{\frac{2DC_0}{C_h}}$$

$$Q_3^* = \sqrt{\frac{2*1200*10}{0.20*10.8}}$$

=105

Since $Q_3^* < b_2$ and $Q_3^* < b_1$

TC(b_1=200)=D.C+TVC

=13968

TC(b_2=1000)=D.C+TVC

=14052

Since TC $(b_1)<TC(b_2)<TC(Q_1^*)$

Hence the shopkeeper accept the offer 5% discount

His net saving per year would be 14640-13968= Rs672

Two price break

1)The annual demand of product is 10000units. Each unit cost RS 100 if the orders placed in quantities below 200 units but for orders of 200 or above the price is Rs95. The annual inventory holding cost is 10% of the value of the item and the ordering cost is rs5 per order. Find the economic lot size.

Solution

D= 10000 units per yr

C_0=5 per order

r=10%

C=100

Quantity	prince/unit
$0\leq Q_1<200$	100
$200\leq Q_2$	95

b_1=200

$$Q_2^* = \sqrt{\frac{2DC_0}{C_h}}$$

$$= \sqrt{\frac{2*10000*5}{0.10*95}}$$

=103

Since $Q_2^* < b_1$

We calculate Q_1^* using $C_1=100$

$$= \sqrt{\frac{2*10000*5}{0.10*100}}$$

=100

$TC(Q_1^*) = D.C_2 + TVC$

$= 1001000$

$TC(b_1^*) = D.C_2 + TVC$

$= 950000$

$TC(b_1=200) = 950000 + 1200$

$= 951200$

$TC(Q_1) > TC(b_1)$

Therefore optimal order quantity is $Q^* = b_1 = 200$ units

UNIT-IV

NETWORK

PROJECT

A project defined as a combination of *inter related activities*, all of which must be executed in a *certain order* to achieve the goal.

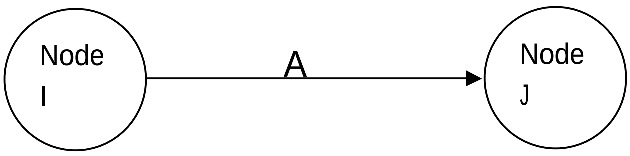
EXAMPLE:-

Construction of building is the project consist of several levels (activities) of works

ACTIVITY

It is a *task* or an *item of work* to be done in a project.

An activity can be represented by an arrow with a node (event) at the beginning and at the end, arrow indicating the termination of activity.



NOTE:-

In the above arrow diagram,

A denotes the *activity.*

Node I is called *starting* node (Node I is also called *tail* event)

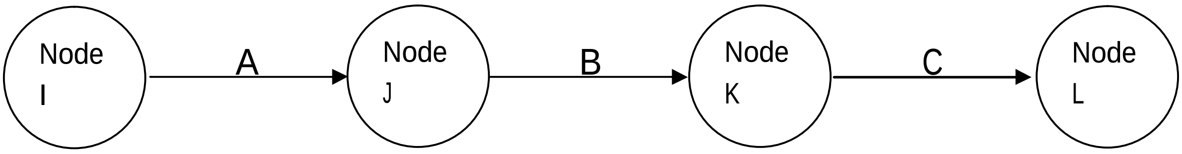
Node II is called *ending* node (Node II is also called *head* event)

Symbolically the above activity can be written as *I < J* .

ARROW DIAGRAM

The diagram in which *arrow represents an activity* is called arrow diagram.

IMMEDIATE PREDECESSOR / SUCCESSOR



In the above arrow diagram,

Activity 'A' is called *immediate predecessor* of the activity B.

Activity 'B' is called the *immediate successor* of the activity A.

Activity 'B' is called *immediate predecessor* of the activity C.

Activity 'C' is called the *immediate successor* of the activity B.

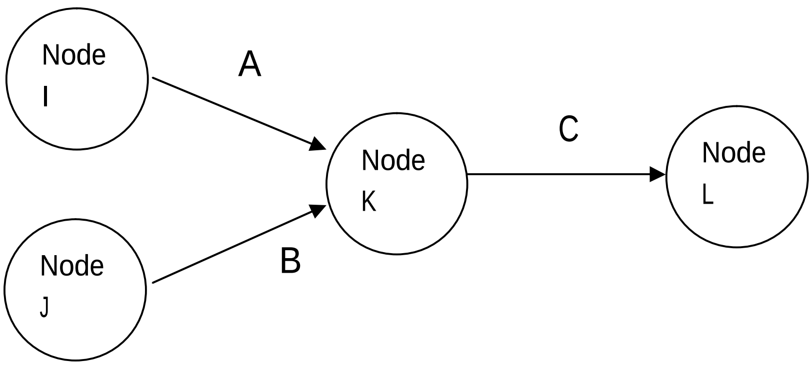
NOTE:-

(i) C is a successor of A, *but not* immediate successor

(ii) A is a predecessor of C, *but not* immediate predecessor.

(iii) It can be written *short form* as follows *A,B<C* (or C>A,B) and *A<B* (or B>A)

EXAMPLE:-



Activity	*Relation*	*Immediate Predecessor*	*Immediate Successor*
A	A<C (C>A)	-	C
B	B<C (C>B)	-	C
C	A,B<C (C>A,B)	A,B	-

STARTING ACTIVITY

An activity which does not have predecessor is called starting activity in the project.

NOTE:-

(i) There may be one or more than one starting activities in the network diagram.

(ii) The starting nodes of all the starting activity can be combined into a single nodes.

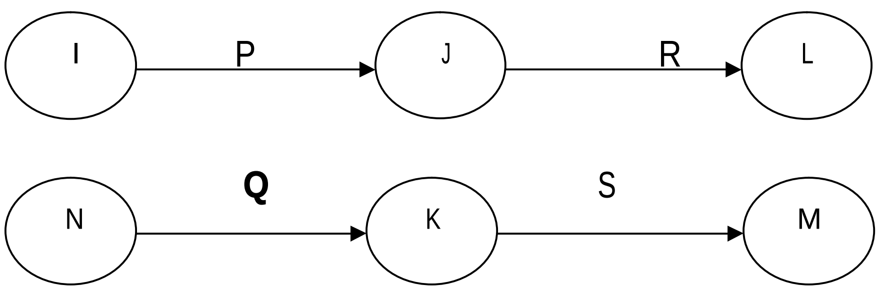
EXAMPLE

From the following construct the arrow diagram.

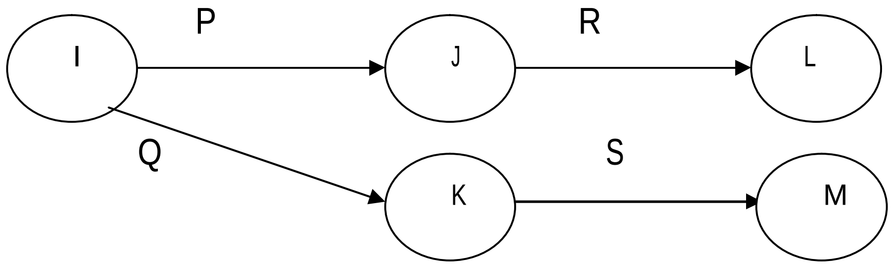
Activities :	P	Q	R	S
Predecessor:	-	-	P	Q

SOLUTION:-

Since P, Q *has no predecessor's*, therefore P and Q are *starting nodes*.



Here P,Q are starting activities, therefore the starting notes of these activities namely I and N can be combined into single node.



ENDING ACTIVITY

An activity which does not have successor is called ending activity in the project.

NOTE:-

(i) There may be one or more than one ending activities in the network diagram.

(ii) The ending nodes of all the ending activity can be combined into a single nodes.

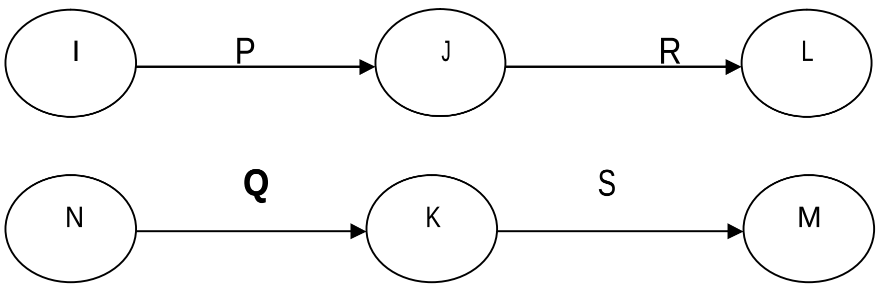
EXAMPLE

From the following construct the arrow diagram.

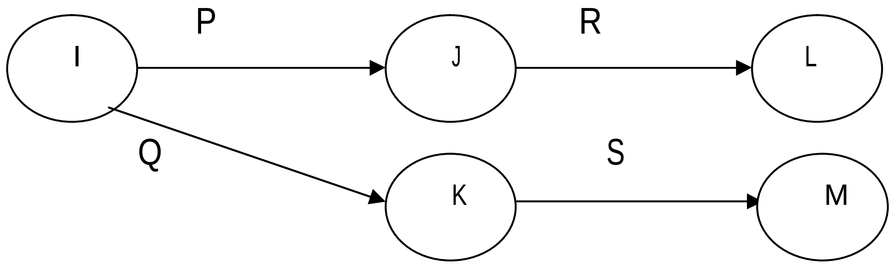
Activities :	P	Q	R	S
Predecessor:	-	-	P	Q

SOLUTION:-

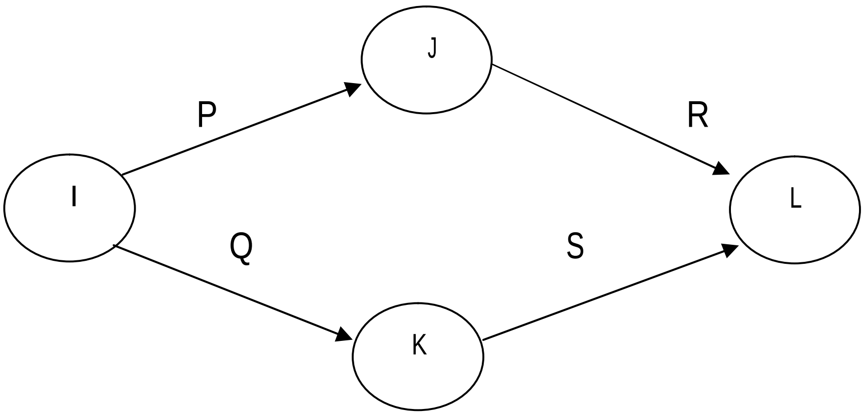
Since P, Q *has no predecessor's*, therefore P and Q are *starting nodes*.



Here P,Q are starting activities, therefore the starting notes of these activities namely I and N can be combined into single node.



Since R,S *has no successor's*, therefore R and S are *ending nodes*, Therefore the ending nodes of the ending activities can be combined into single node



DUMMY ACTIVITY

If the project contains two or more activities which have some of their *immediate predecessor in common* then there is a *need for introducing* dummy activity.

EXAMPLE

From the following construct the arrow diagram.

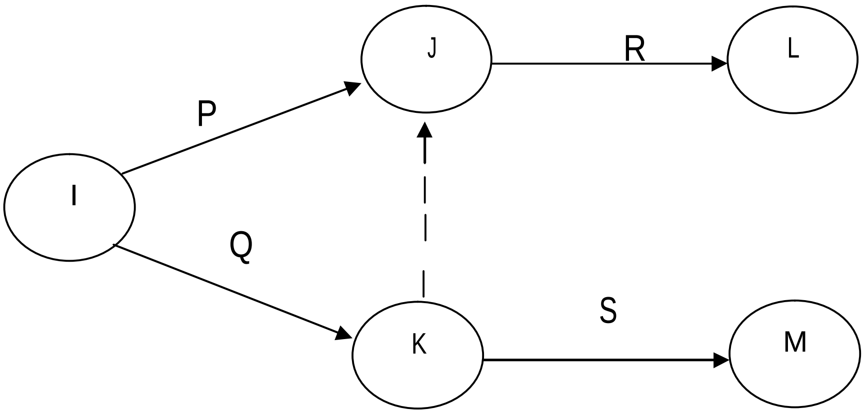
Activities :	P	Q	R	S
Predecessor:	-	-	P,Q	Q

SOLUTION:-

Since P, Q *has no predecessor's*, therefore P and Q are *starting nodes*.

Similarly R, S *has no successor's*, therefore R and S are *ending nodes*.

Here activities R and S have *predecessor Q in common*, therefore we must *add dummy activity* in the network diagram. Otherwise we could not draw the arrow diagram. At the same time P is not the predecessor of S.



EXAMPLE:-

Construct the arrow diagram for the following

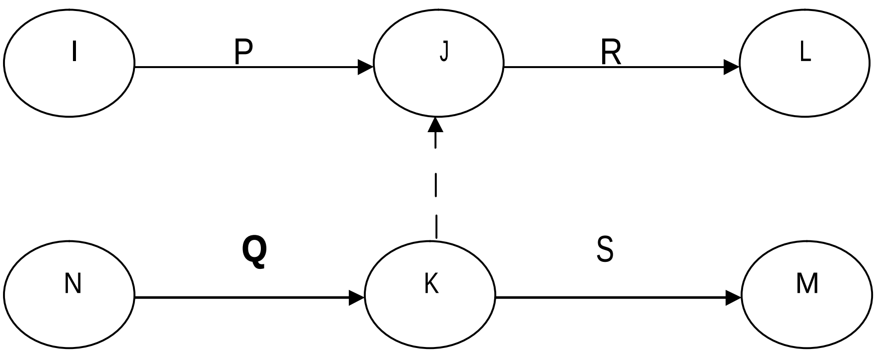
Activity	Predecessor
P	-
Q	-
R	P,Q
S	Q

SOLUTION:-

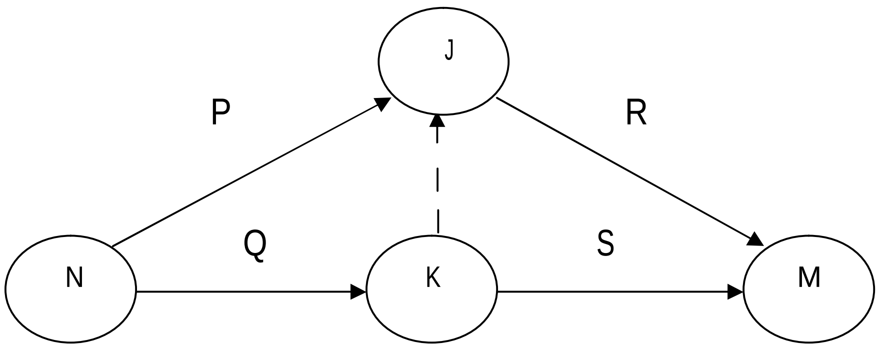
Since P,Q has no predecessor, therefore P,Q are starting activities

Similarly R,S has no successor, there for R,S are ending activities.

Here the activity Q is the common predecessor for both the activities R and S, So there is a need of dummy activity. (without adding dummy activity the arrow diagram could not be completed)



The starting and ending nodes of starting and ending activities can be combined into single node.



NOTE:-

(i) Dummy activity is an *imaginary activity* which *does not consume* any resources and it serves the purpose of indicating the predecessor or successor.

(ii) Dummy activity is represented by a *dotted line* in the arrow diagram.

(iii) When there are two activities with common predecessor's, then there *may be a need of adding dummy activity* , otherwise no need to add. It should be added only when the arrow diagram could not be completed without this.

RULES FOR CONSTRUCTING PROJECT NETWORK

(i) There must be no loop

(ii) Only one activity should connect any two nodes.

(iii) No dangling should appear in a project.

i.e. No ending node of any activity, expect terminal activities of the project should be left without any activity emanating from it.

RULES FOR NUMBERING OF NODE

(Foral and Fulkerson's algorithm or rule)

Step (i) Number the start node which has no predecessor activity as 1.

Step (ii) Delete all the activities emanating from this node 1, number all the resulting start node without any predecessor as 2,3.....

Step (iii) Delete all the activity originating or emanating from 2,3,.....

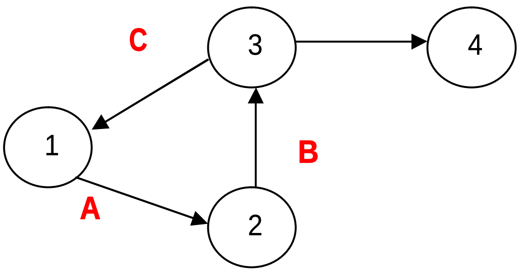
Step (iv) Number all the resulting new starting nodes left in step 3.

Step (v) Repeat the process until the terminal node without any successor activity is reached and number the terminal node suitably.

ERRORS IN NETWORK

(i) Looping

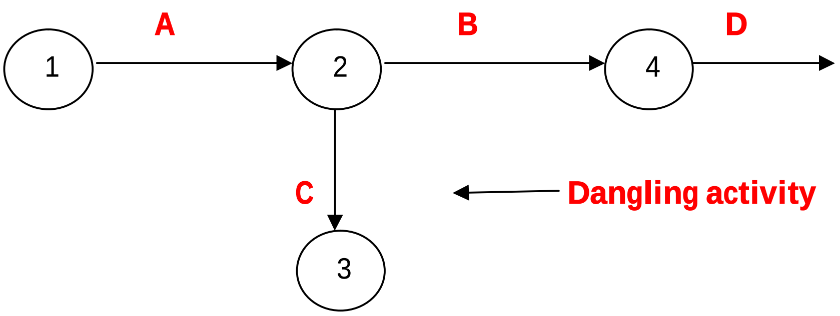
A case of endless loop in a network diagram which also known and looping. Looping is considered as faults in a network, therefore it must be avoided.



(Looping not allowed)

(ii) Dangling

A case of disconnect activity before the completion of all activities which is also known as dangling. Dangling is considered as faults in network, therefore it must be avoided.

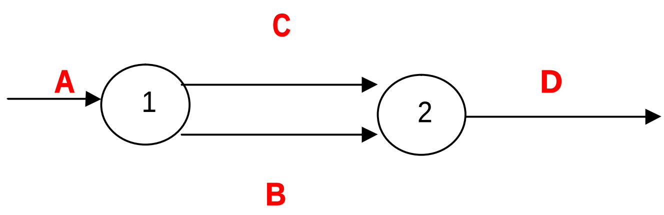


Here C is a dangling activity, since the activity is disconnected before completion of the activity D.

ADDITION OF DUMMY ACTIVITY IN NETWORK

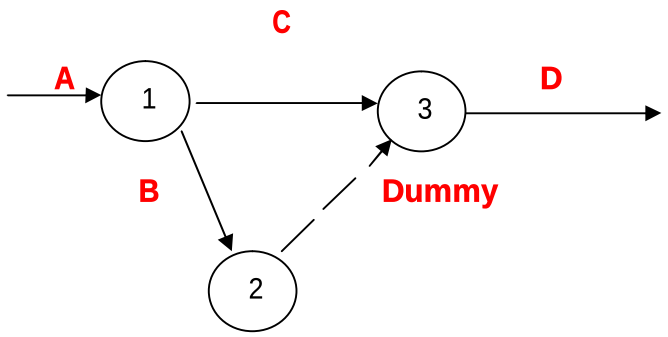
There are two situations in which use of dummy activity may help in drawing the network correctly as follows

(I) When two or more parallel activities in a project have the same head and tail events. i.e two events are connected with more than one arrow. These parallel activities are not allowed in network.

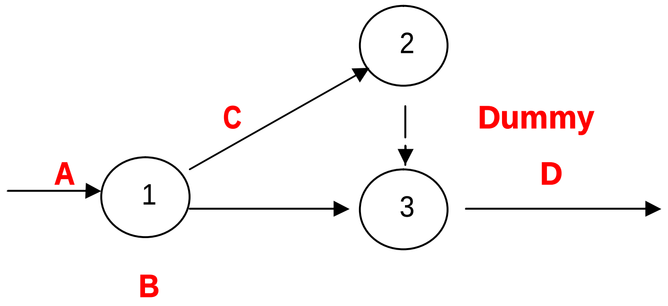


Here the activities B and C have common predecessor A and successor D.

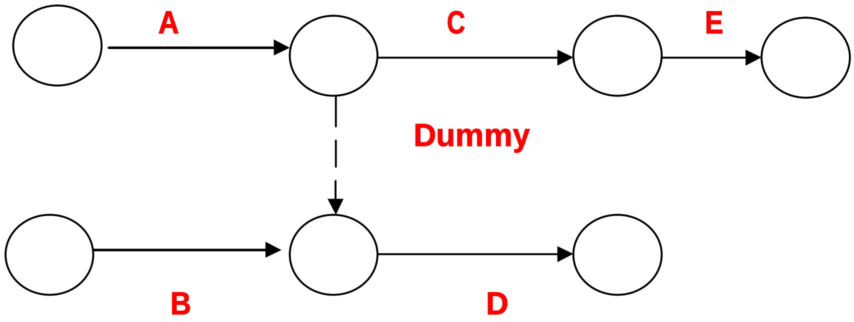
This can be corrected using dummy activity as follow



Otherwise



(II) When two chains of activities have a common predecessor, then we may have to add dummy.



Here C and D has common predecessor A, but B is not a predecessor of C.

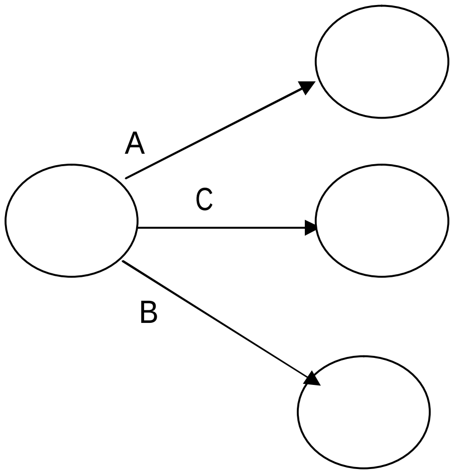
PROBLEM:-01

Draw the network for the project whose activities which predecessor activities are given below.

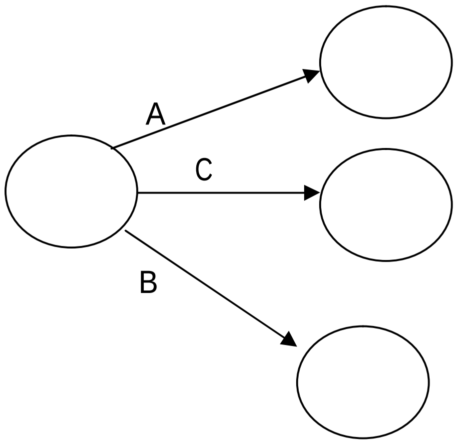
A,B,C- can start simultaneously . E>B,C; F,G>D; H,I>E,F; J>I,G; K>H; D>A.

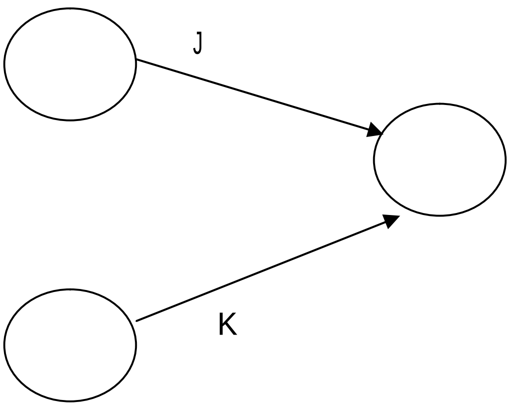
SOLUTION:-

Since there is no predecessor's for the activity A,B,C, therefore A,B,C are the starting activity

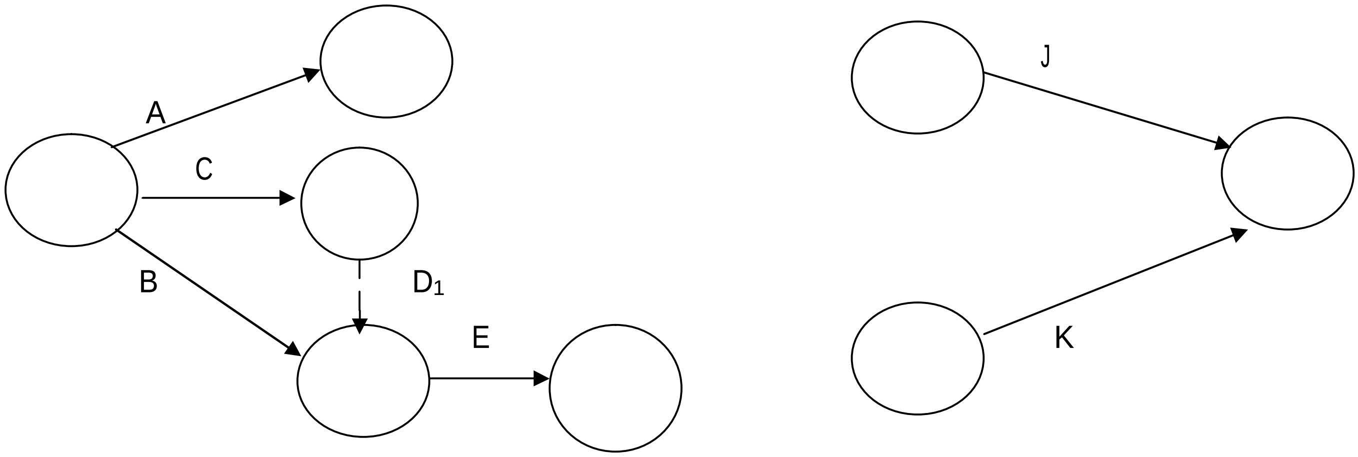


Similarly there is no successor's for the activities J,K, therefore J,K are ending activities.

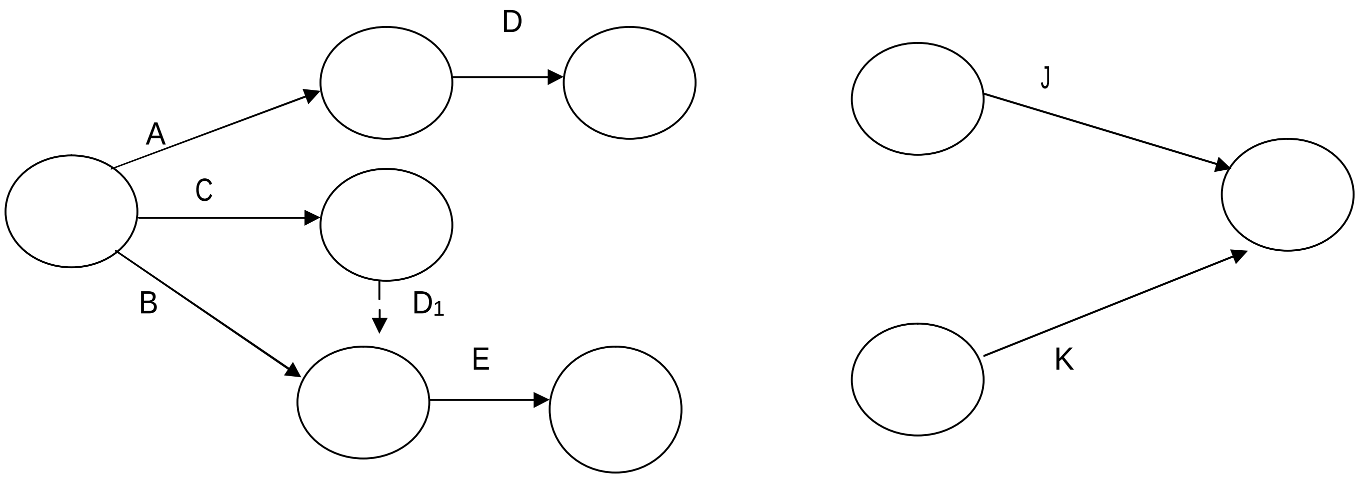




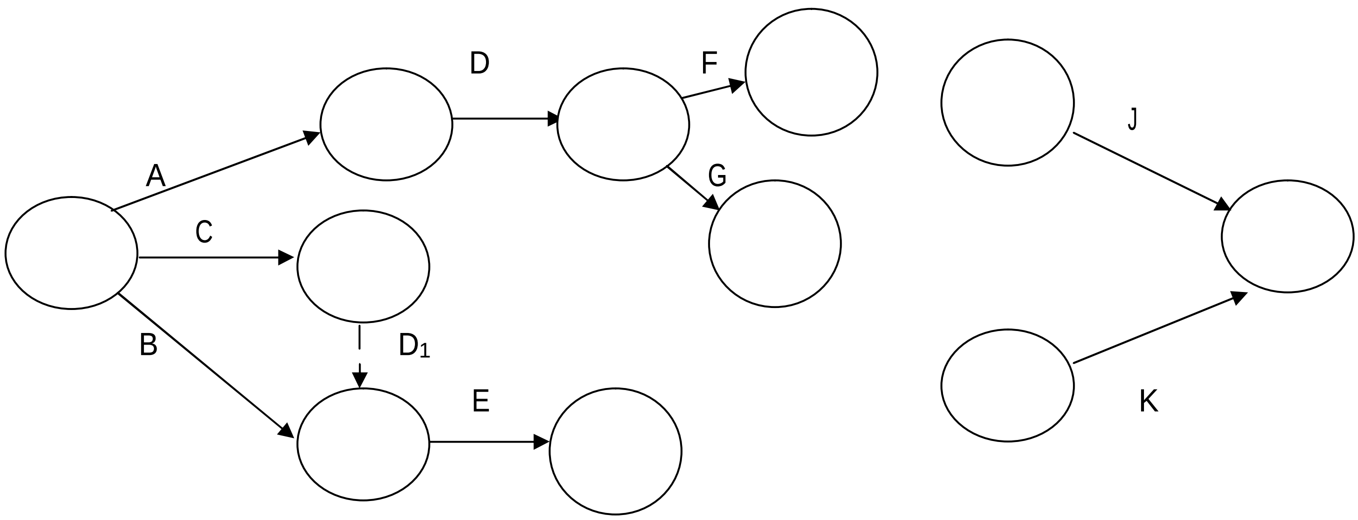
Since E>B,C, i.e E is the successor of B and C.



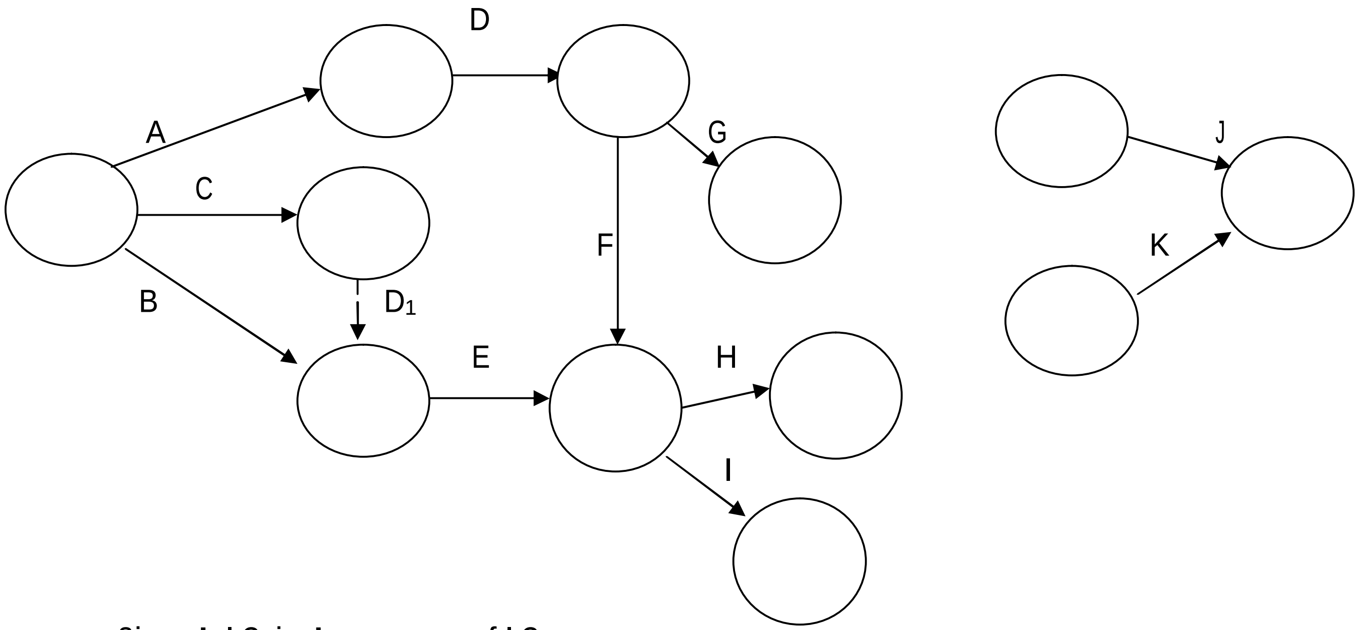
Since D>A, i.e D is the successor of A.



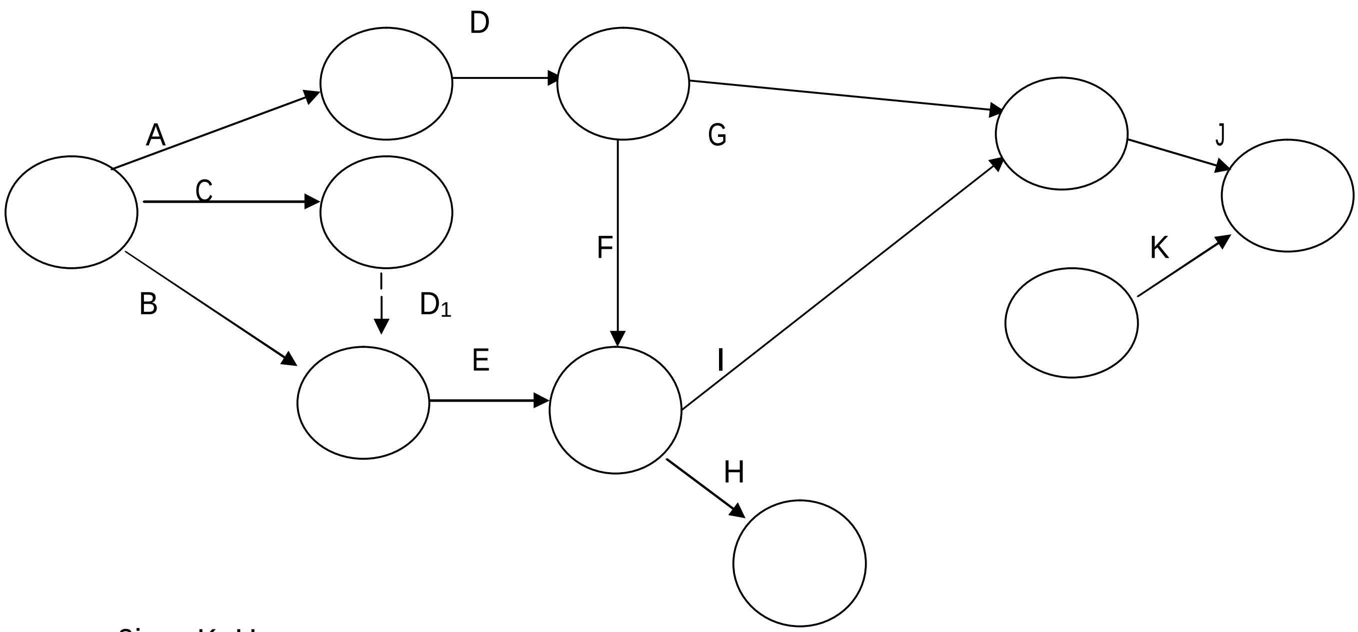
Since F,G>D, i.e F,G is the successor of D



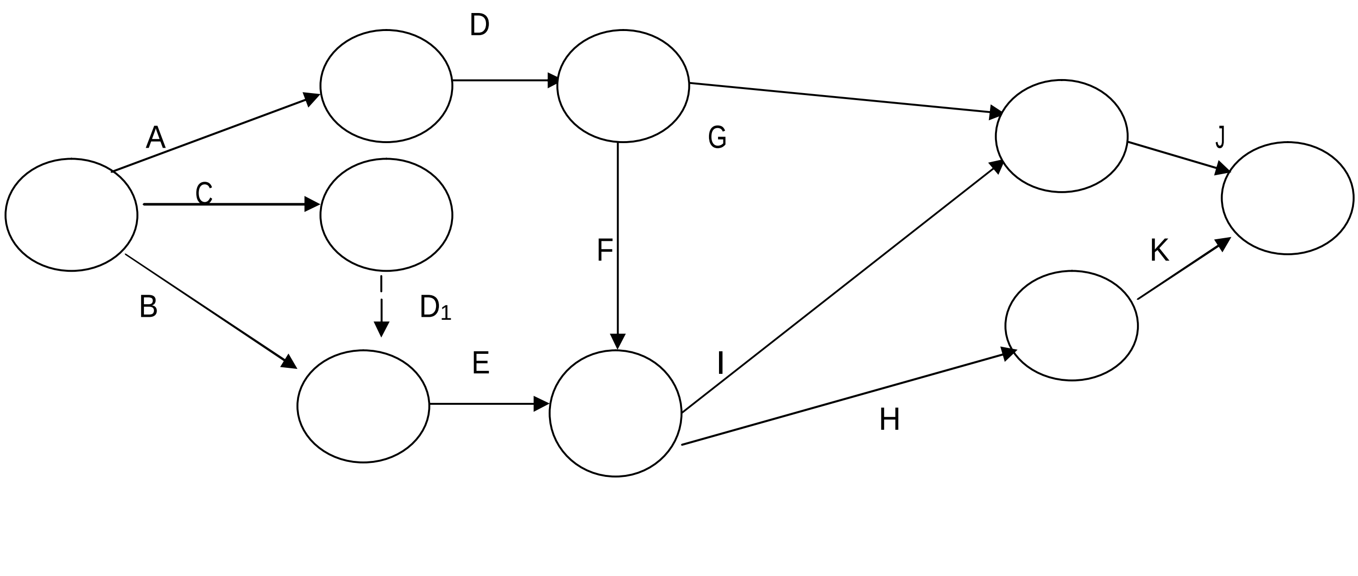
Since H,I>E,F



Since J>I,G, i.e J successor of I,G



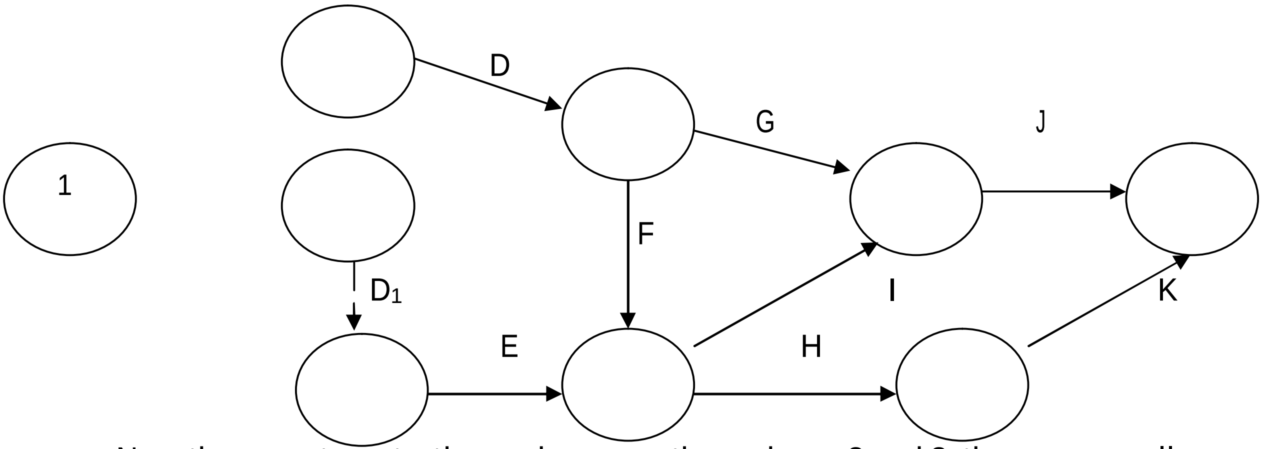
Since K>H



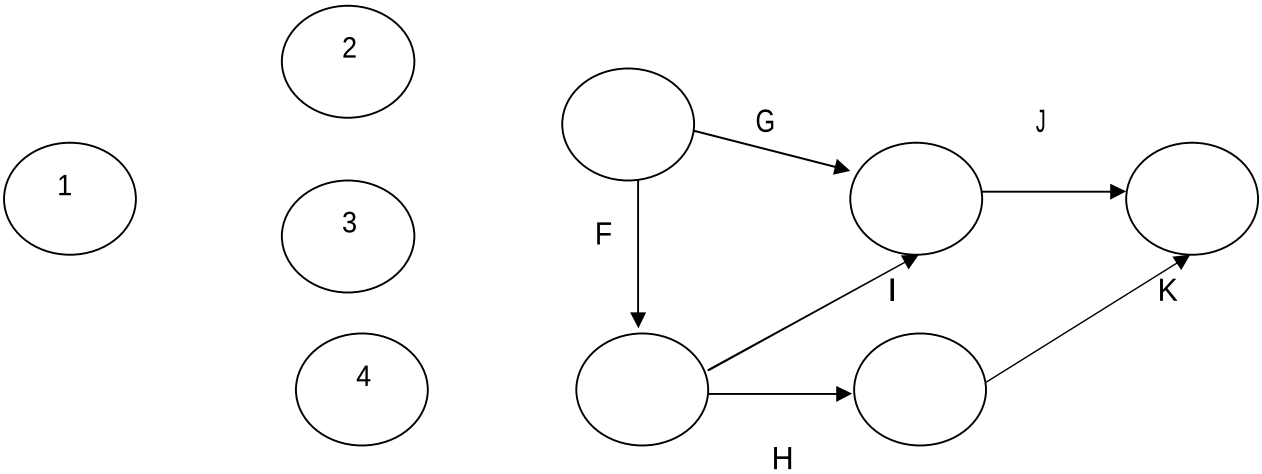
NOTE:-

Numbering of nodes

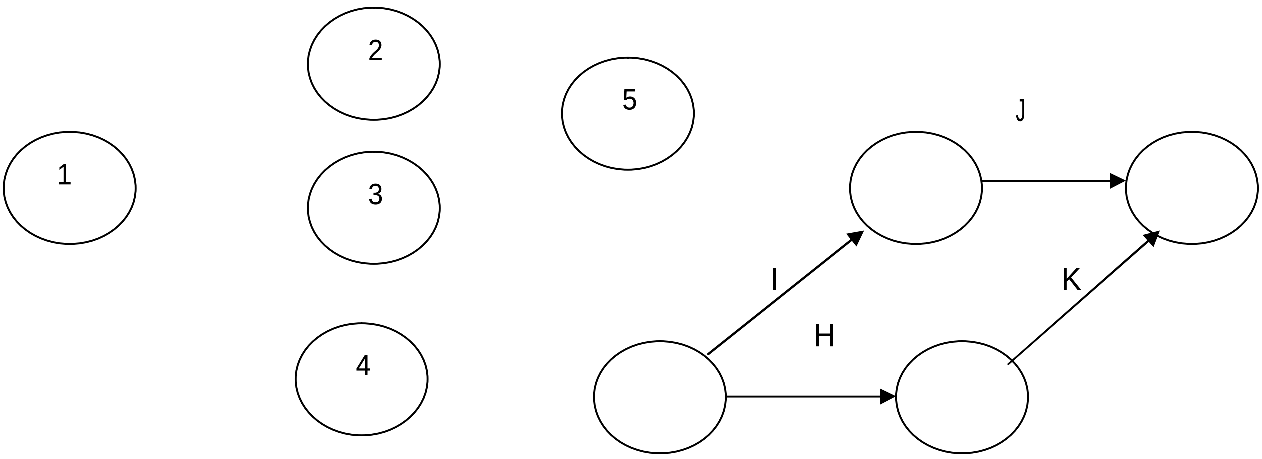
First we have to number the starting node as 1, then remove all activities emanating from the node 1, namely A, B and C



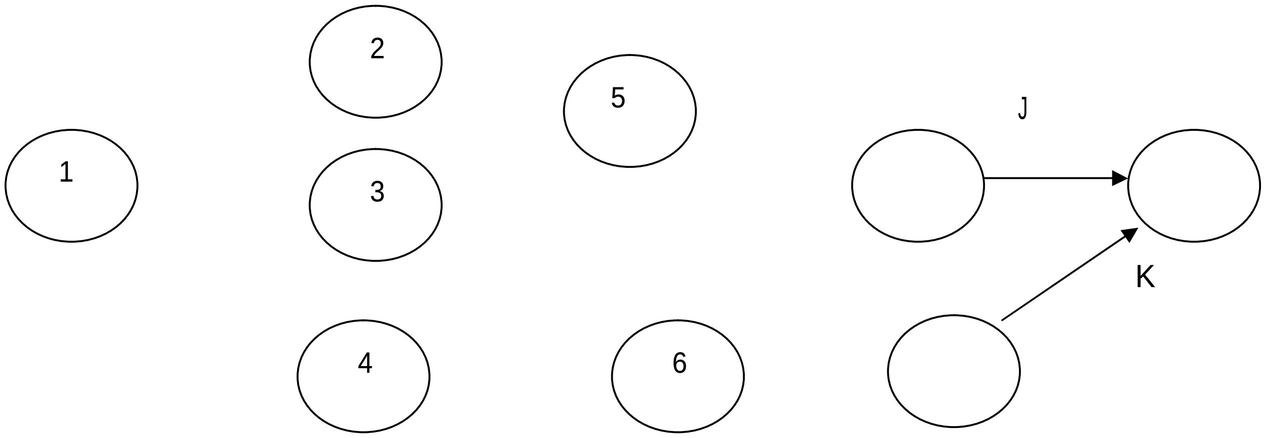
Now there are two starting nodes, name the nodes as 2 and 3, then remove all activities emanating from node 2 and node 3. Namely D , D_1 and E



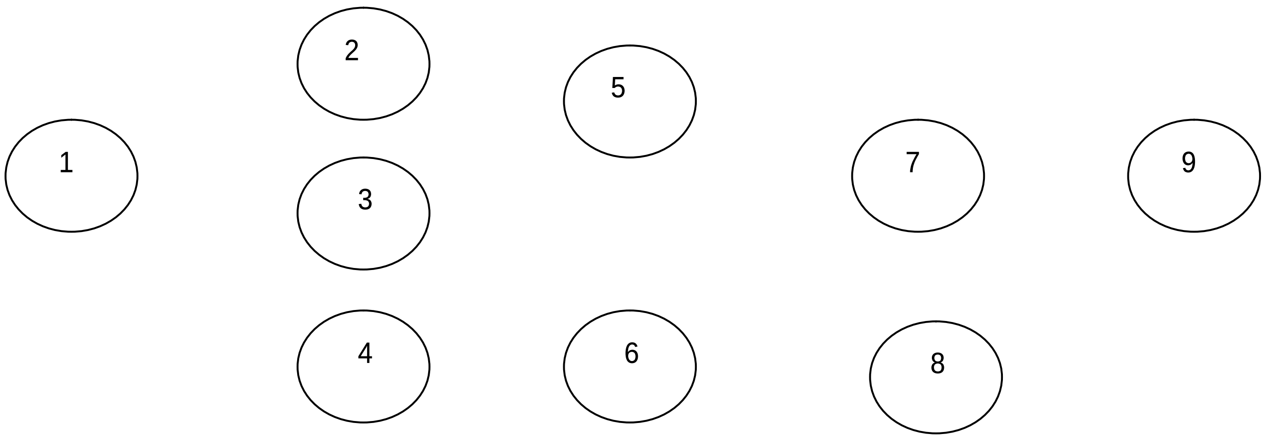
Now there are two starting node, name it as 4 and 5, remove all activities emanating from node 4 and node 5. Namely F and G.



Now there is only one starting node, name it as 6, remove all activities emanating from node 6. Namely I and H.



Now there is two starting nodes, name it as 7 and 8, remove all activities emanating from node 7 and 8. Namely J and K.



Name the final node as 9.

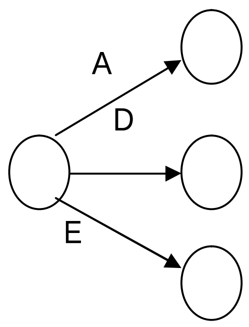
PROBLEM:-02

Construct the network for the project whose activities and their relation are given below:

A,D,E can start simultaneously. B,C>A; G,F>D,C; H>E,F

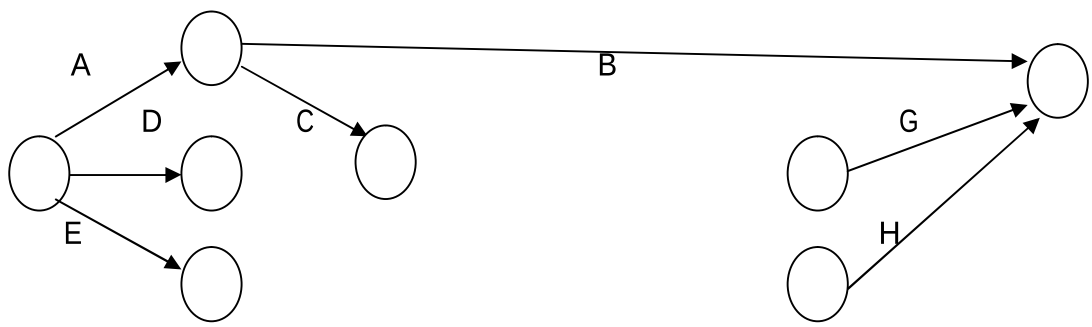
SOLUTION:-

Since there is no predecessor's for the activities A, D and E , therefore starting activities are A,D,E

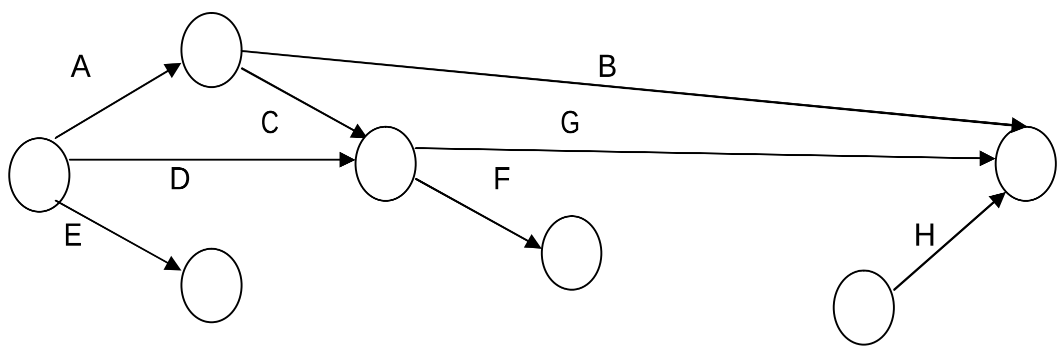


Since there is no successors for the activities B, G and H, therefore the B,G,H are ending activities

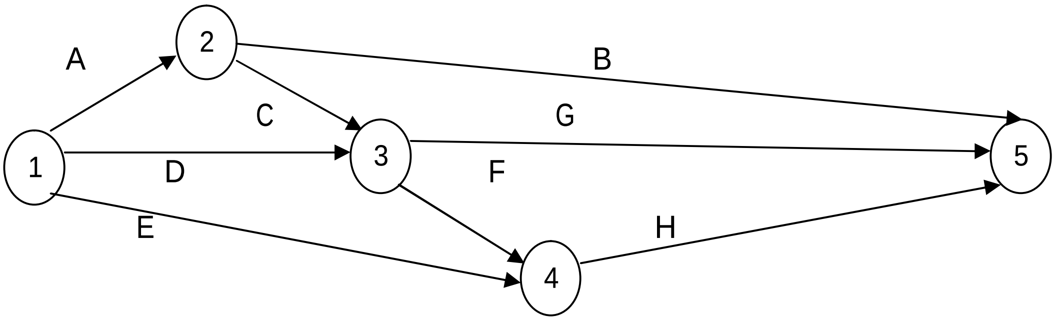
Since B,C>A, i.e B and C are successor's of A



Since G,F>D,C, i.e G and F are Successors for the activities D and C.



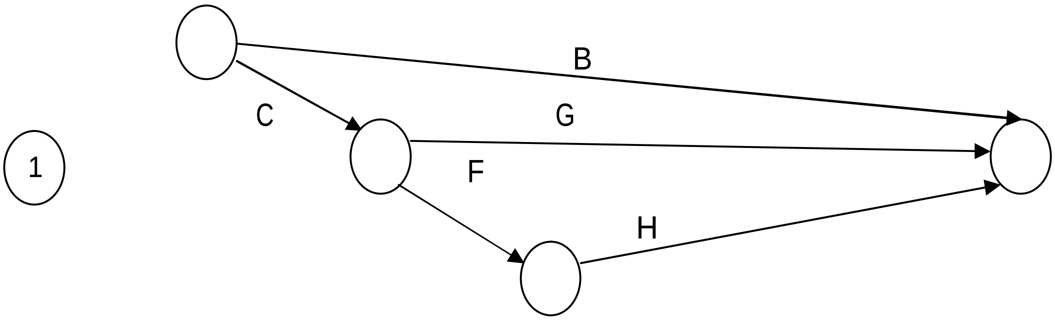
Since H>E,F, i.e H is the successor for the activities E and F.



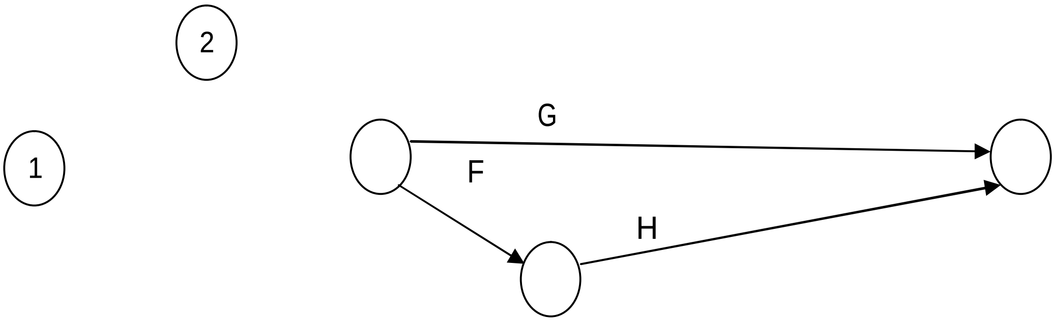
NOTE:-

Numbering of nodes

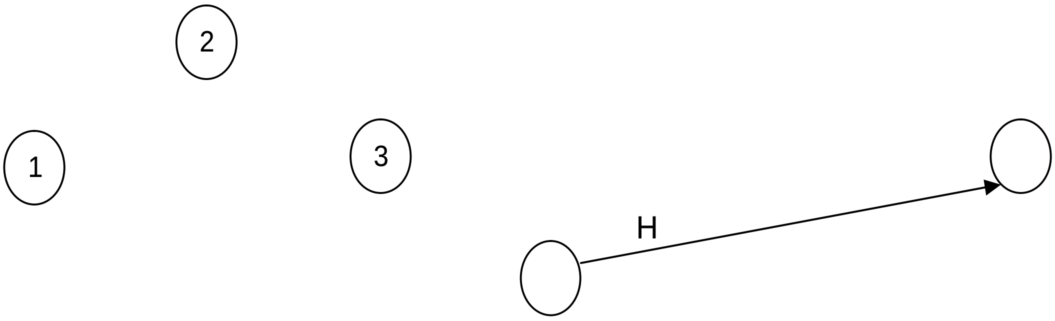
First we have number initial node as 1, then remove all activities emanating from node 1. namely A , D and E.



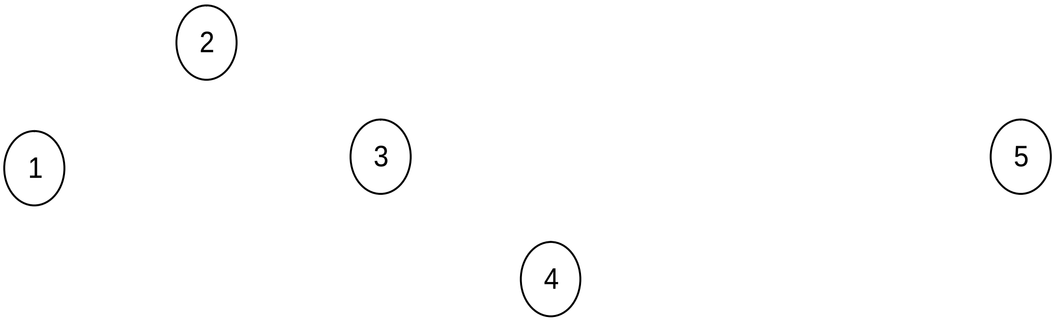
Now there is only one starting node, name it as 2, remove edges emanating from node 2. namely B and C.



Now there is only one starting node, name it as 3, remove edges emanating from node 3 . namely G and F.



Now there is only one starting node, name it as 4, remove edges emanating from node 4 . namely H.



Name the final node as 5.

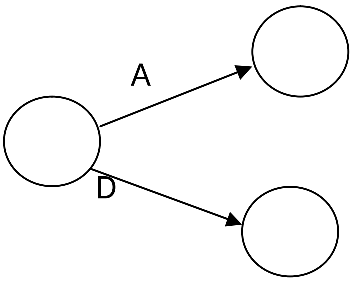
PROBLEM:-03

Draw the network for the project whose activities and their relationship are given below.

ACTIVITY	A	B	C	D	E	F	G	H	I
IMMEDIATE PREDECESSOR	-	A	A	-	D	B,C,E	F	E	G,H

SOLUTION:-

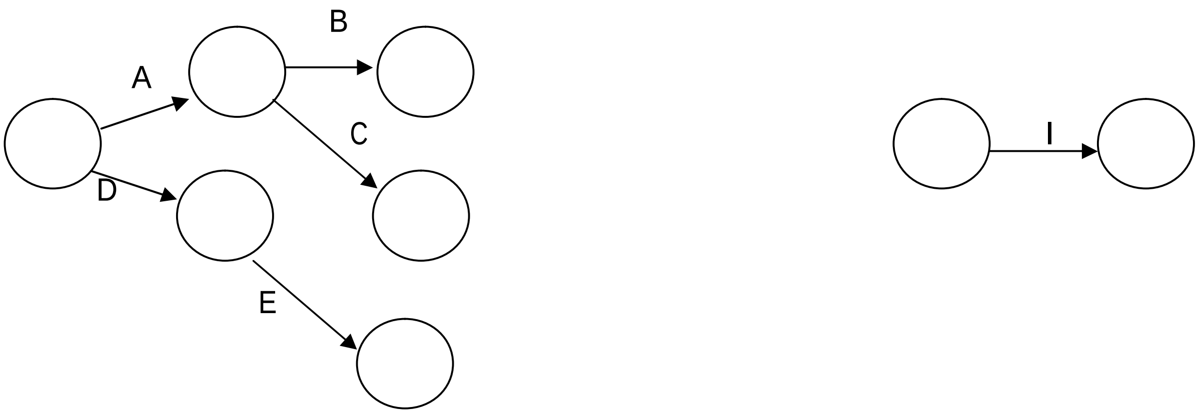
Since there is no predecessor's for the activities A and D, therefore A,D are the starting activities.



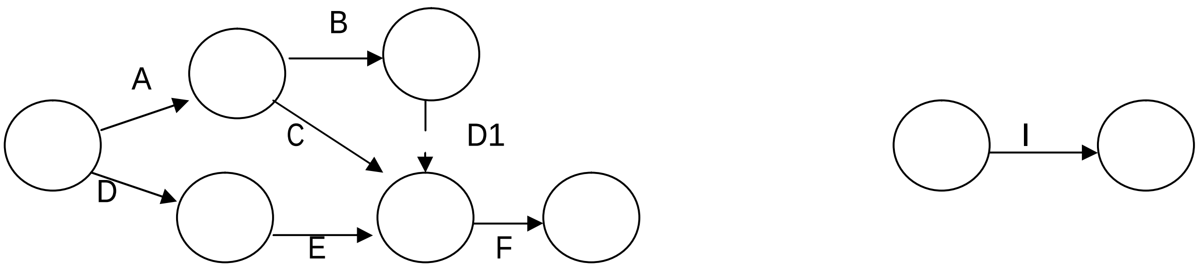
Since there is no successor's for the activity I, therefore I is the ending activity.

Since A is predecessor for activities B and C

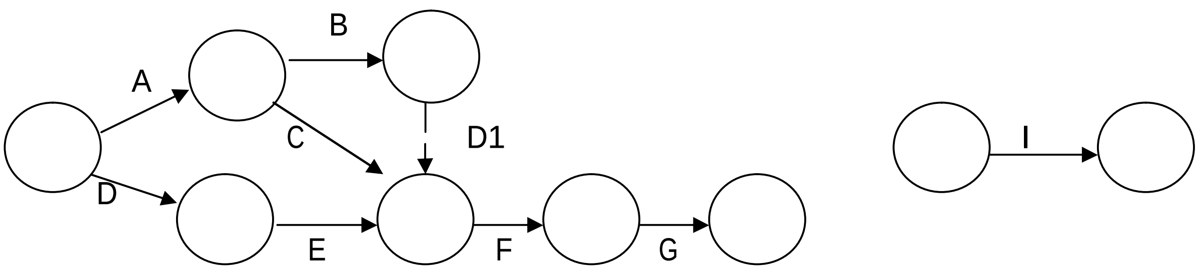
Since D is predecessor for activity E



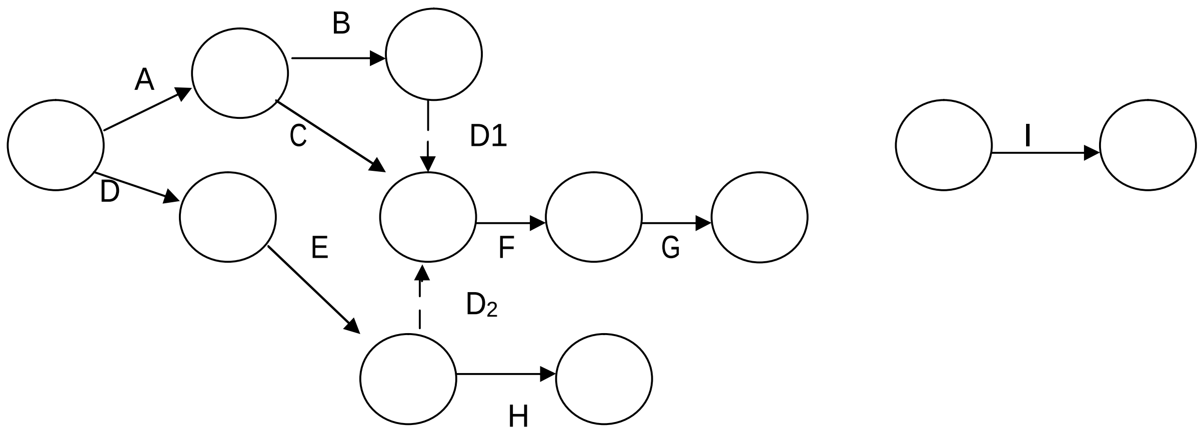
Since B,C and E are predecessor for activity F



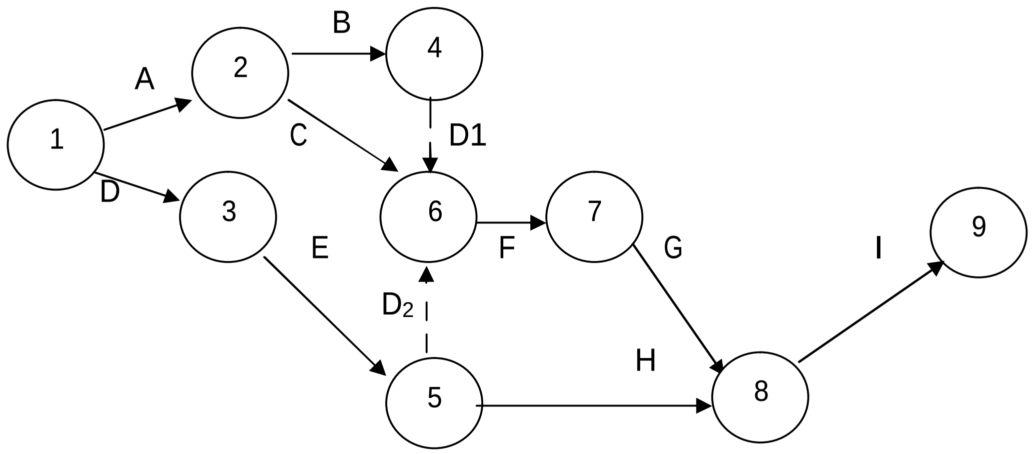
Since F is predecessor for activity G



Since E is predecessor for activity H, we have to add dummy activity(D_1), since two activities F and H are having predecessor E in common. Also the activities B, C are predecessor's for the activity F but not for the activity H.



Since G, H are the predecessor for activity I.



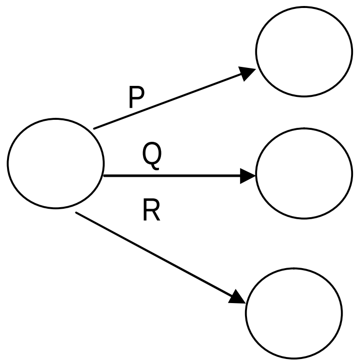
PROBLEM:-04

Draw the network for the project whose activities and their relationship are given below.

P	Q	R	S	T	U
-	-	-	P,Q	P,R	Q,R

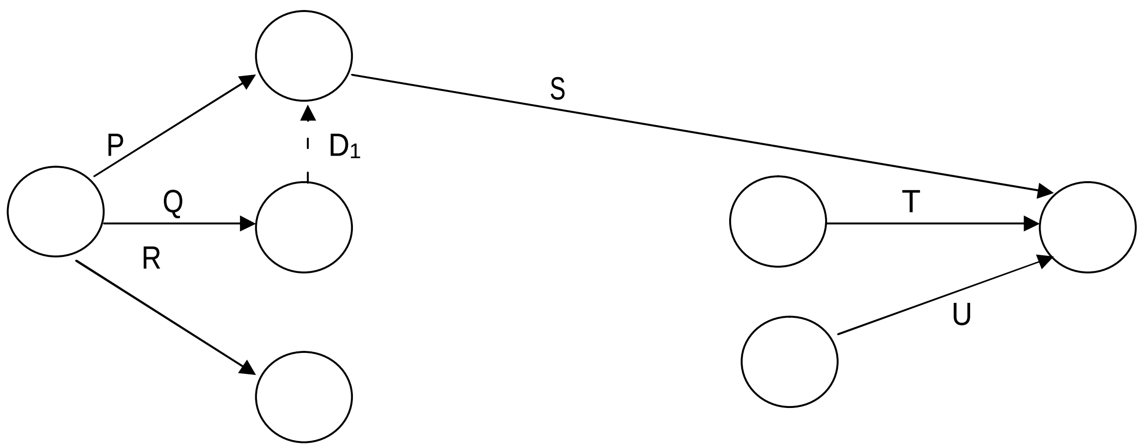
SOLUTION:-

Since there is no predecessor's for the activities P,Q and R, therefore P,Q,R are starting activities.

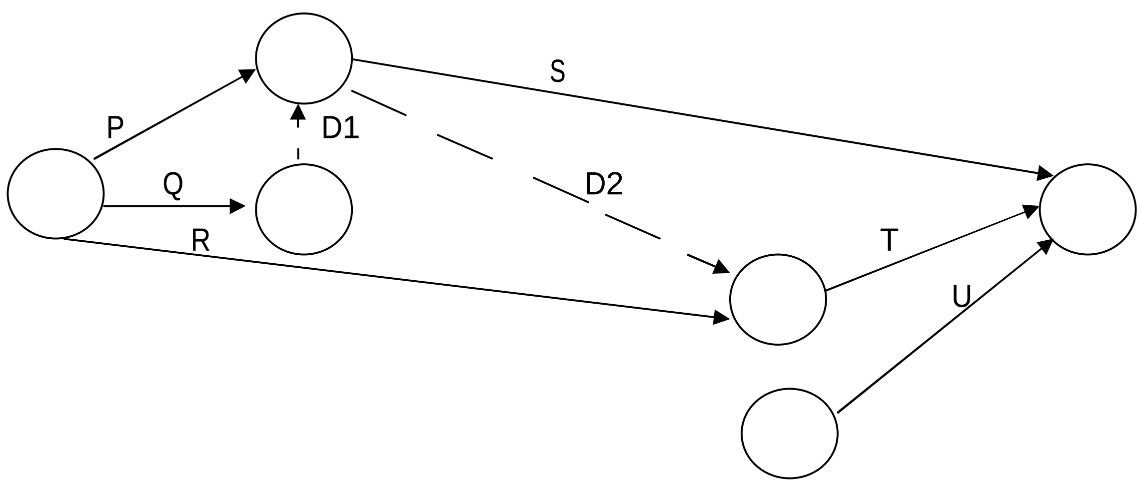


Since there is no successor's for the activities S,T and U, therefore S,T and U are ending activities.

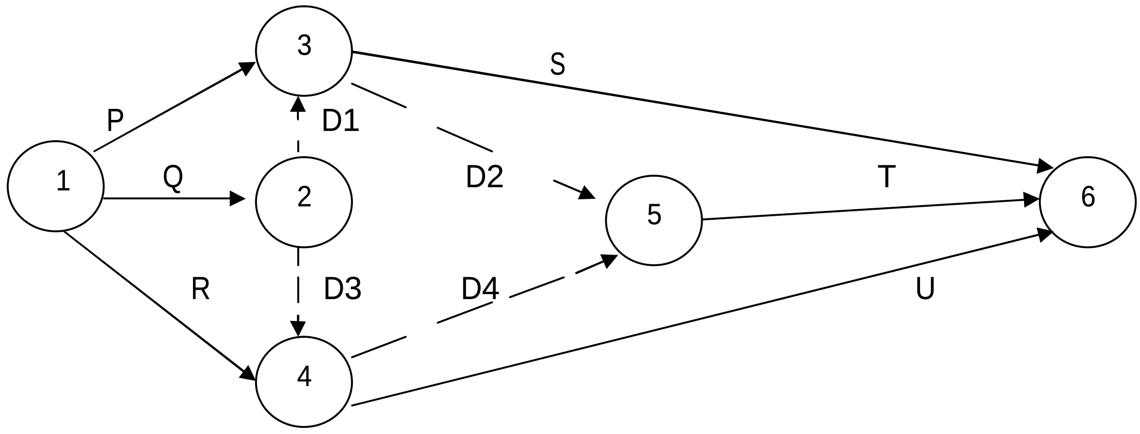
Since P,Q are the predecessor for S, we have to add dummy activity D_1



Since P,R are the predecessor for T. We have to add dummy activity [D_2], since two activities S and T have predecessor P in common.



Since Q,R are the predecessor for U. We have to add dummy activities[D_3 and D_4], since two activities U and T have predecessor R in common.



EXERCISE

1. Draw network diagram from following activities and find critical path and total slack of

activities.

Job	A	B	C	D	E	F	G	H	I	J	K
Immediate Predecessor	-	A	B	C	B	E	D,F	E	H	G,I J	

2. A small project is composed of 7 activities whose time estimates are listed in the table below.

Activity	A	B	C	D	E	F	G	H
Predecessor	-	-	-	A	B	C	D,E	F,G

Draw the network diagram.

3. A small project consists of seven activities for which the relevant data are given below;

Activity	A	B	C	D	E	F	G
Preceding Activity	-	-	-	A,B	A,B	C,D,E	C,D,E

Draw the network diagram.

4. Draw the network for the following

Activity	Predecessors
A	-
B	-
C	A
D	B
E	B
F	C,D
G	E

5. A Professional institute is organizing a seminar on new accounting techniques at Delhi.

The seminar will include 2 keynote speeches and 8 paper reading sessions. It is considered that the following activities are to be accomplished with the respective duration as specified in the table below;

Activity	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Preceding	-	-	-	B	B	A,E	C,D	G	H	I	C,D	C,D	C,D	J	M

6. A small project consists of seven activities, the details of which are given below:

Activity	Immediate predecessor
A	-
B	A
C	A
D	B,C
E	B
F	D,E
G	D

Draw the network diagram.

PERT and CPM

PERT (Programme Evaluation Review Technique) and CPM (Critical Path Method) are network techniques used for *planning, scheduling* and *executing* large projects which require coordination and execution of large number of activities. These activities should be completed within a specified *time, cost* and *meeting* the performance standards.

The *objective* of PERT and CPM is to estimate the *total project duration* and to assign *starting and finishing times* to all activities involved in the project. This help in *checking actual progress* against the scheduled duration of the project.

The duration of individual activities may be *uniquely determined* in case of CPM , But it may involve the *three time estimate* in case of PERT out of which the expected duration of an activity is computed.

NOTATIONS

The following are the notations used in the PERT analysis.

E_i = Earliest occurrence time of an event i, It is earliest time at which an event occur without affecting the total project time.

L_i = Latest occurrence time of an event i, It is the latest time at which an event can occur without affecting the total project time.

ES_{ij} = Earliest start time for an activity (i,j). It is the earliest time at which the activity can start without affecting the total project time.

LS_{ij} = Latest start time for an activity (i,j). It is the latest possible time by which the activity must start without affecting the total project time.

EF_{ij} = Earliest finish time for an activity (i,j). It is the earliest possible time at which an activity can finish without affecting the total project time.

LF_{ij} = Latest finish time for an activity (i,j). It is the latest time by which an activity must get completed without delaying completion. without affecting the total project time.

t_{ij} = Duration of an activity (i,j)

NOTE:

(I) In a network diagram there should be *only* one initial event and one end event.

(II) For the calculation of earliest start, finish, and latest start, finish time. we use the *forward pass method* and *backward pass method* respectively.

FORWARD PASS METHOD

(To calculate earliest start and finish)

The calculations of forward pass method begin from the initial event (Node 1), proceed through the events in an increasing order of event numbers and end at the final (Node N)

Step:-01

Let the earliest occurrence time of initial event 1 be zero. *i.e* $E_1=0$

Step:-02

Calculate the earliest start time (ES_{ij}) for each activity (i,j) , where i=1

$ES_{ij} = E_i$, for all activity (i,j) starting at event i.

Step:-03

Calculate the earliest finish time (EF_{ij}) of each activity (i,j) that begin at event i.

$EF_{ij} = ES_{ij} + t_{ij}=E_i+t_{ij}$ for all activity (i,j) beginning at event i

Step:-04

Proceed to the next event J where j>i

Step:-05

Calculate the earliest occurrence time for the even j, This is the maximum of the earliest finish times of all activities ending into that event.

$E_j=Max\{EF_{ij}\}=Max\{E_i+t_{ij}\}$ for all immediate predecessor activities.

Step:06

If j=N (final event), then earliest finish time for the project, that is earliest occurrence time E_N for the final event is given by

$E_N=Max\{EF_{ij}\}=Max\{E_{N-1}+t_{ij}\}$, for all terminal activities.

BACKWARD PASS METHOD

(To calculate latest finish and start)

The calculations of backward pass method begin from the end event (Node N), proceed through the events in an decreasing order of event numbers and end at the initial (Node 1)

Step:-01

Let the latest occurrence time of last event N equal to its earliest occurrence time $L_N=E_N$ (known from forward pass method)

Step:-02

Calculate the latest finish time (LF_{ij}) for each activity (i,j), which ends at the event j.

$LF_{ij} = L_j$, for all activity (i,j) ending at event j.

Step:-03

Calculate the latest start time (LS_{ij}) of each activity (i,j) that end at event i.

$LS_{ij} = LF_{ij} - t_{ij}=L_j-t_{ij}$ for all activity (i,j) ending at event j

Step:-04

Proceed to backward go to the previous event, i < j

Step:-05

Calculate the latest occurrence time for the even i, This is the minimum of the latest start times of all activities from the event.

$L_i=Min \{LS_{ij}\}=Min \{L_j-t_{ij}\}$ for all immediate successor activities.

Step:06

If j=1 (initial event), then latest finish time for the project, that is latest occurrence time L_1 for the inital event is given by

$L_1=Min\{LS_{ij}\}=Min\{L_{j-1} - t_{ij}\}$, for all starting activities

CRITICAL ACTIVITY

Certain activities in the network diagram of a project are called critical activities because delay in their executions will cause *further delay* in the project completions time.

FLOAT OF AN EVENT (SLACK)

The float (slack) or free time of the event is difference between its latest occurrence time (L_i) and its earliest occurrence time (E_i).

Event float $=L_i-E_i$

It is a measure of how much later than a particular event could be started without delaying the completion of the entire project.

FLOAT OF AN ACTIVITY (SLACK)

The float (slack) or free time of the activity is the length of time to which a non-critical activity or event can be delayed or extended without delaying the total project completion time.

There are three types of float, **(i)** Total float **(ii)** free float , and **(iii)** Independent float**.** **(iv)** Interfering float

TOTAL FLOAT

It is an amount of time by which an activity can be delayed without affecting project completion time.

Total float for the activity (i,j) is denoted by TF_{ij}

$TF_{ij}=L_j-E_i-t_{ij}$

NOTE

The activity which has total float equal to zero are called as critical activities.

FREE FLOAT

It is calculated to know how much an activity's completion time may be delayed without causing any delay in its immediate successor activities. i.e a delay in performing an activity without affecting floats of subsequent activities.

It is denoted by FF_{ij}

$FF_{ij}=E_j-E_i-t_{ij}$

INDEPENDENT FLOAT

It is the amount of time an activity can be delayed for start without affecting the completion of preceding activities.

It is denoted by IF_{ij}

$IF_{ij}=E_j-L_i-t_{ij}$

NOTE:-

- **(i)** $L_i \geq E_i$ for all i
- **(ii)** Independent float ≤ Free float ≤ Total float.
- **(iii)** Calculations of various float can help the decision -maker in identifying the under utilized resources, flexibility in the total schedule and possibilities of redeployment of recourses

CRITICAL PATH

It is a continuous chain of critical activities in a network diagram. It is the *longest* path starting from first to the last event .

It is usually represented by *double line or think line* in the network diagram.

NOTE:-

The critical path on the network diagram can be identified as follows

(i) For all activity (i,j) lying on the critical path, $E_i=L_i$ and $E_j=L_j$

(ii) On the critical path $E_j-E_i=L_j-L_i=t_{ij}$

LENGTH OF CRITICAL PATH

It is the sum of the individual durations of all the critical activities lying on the critical path and defines the *minimum time* required to complete the project .

THREE TIME ESTIMATE IN PERT

If the duration of activities in project is uncertain, then activity scheduling calculation are done by using the expected value of the durations. Thus rather than estimating directly the expected completion time of an activity, three values are considered. From these timings a single value is estimated for future consideration. This is called three -time estimates in PERT. The three time estimates namely *optimistic, pessimistic* and *most likely times.*

OPTIMISTIC TIME

This is the *shortest possible time* to perform an activity, assuming that every thing *goes well*.

It is denoted by t_o or a

PESSIMISTIC TIME

This is the *maximum time* that is required to perform an activity, under *extremely bad conditions*. However, such conditions do not include *acts of god like earthquakes, flood, etc.,*

It is denoted by t_p or b

MOST LIKELY TIME

This is the *most realistic time* to complete the activity. Statistically, it is the *modal value* of duration of the activity.

It is denoted by t_m or m

NOTE:-

Formulae for PERT Calculations

(1) Expected Activity Time=t_e=$\frac{t_0 + 4t_m + t_p}{6}$

(2) S.D=$\sigma = \frac{1}{6}(t_p - t_o)$

(3) Variance=$\sigma^2 = \left[\frac{1}{6}(t_p - t_o)\right]^2$

(4) Estimation of Project Completion Time:

Probability of completing the project by scheduled time (T_S) can be determined using the standard normal variable value,

i.e P{$T_S \leq r$} can be calculated using Z=$\frac{T_S - T_e}{\sigma_e}$, where T_e is the expected completion time of the project (Total duration of critical path) and σ_e is the number of standard deviations the scheduled time lies from the expected (mean) time.

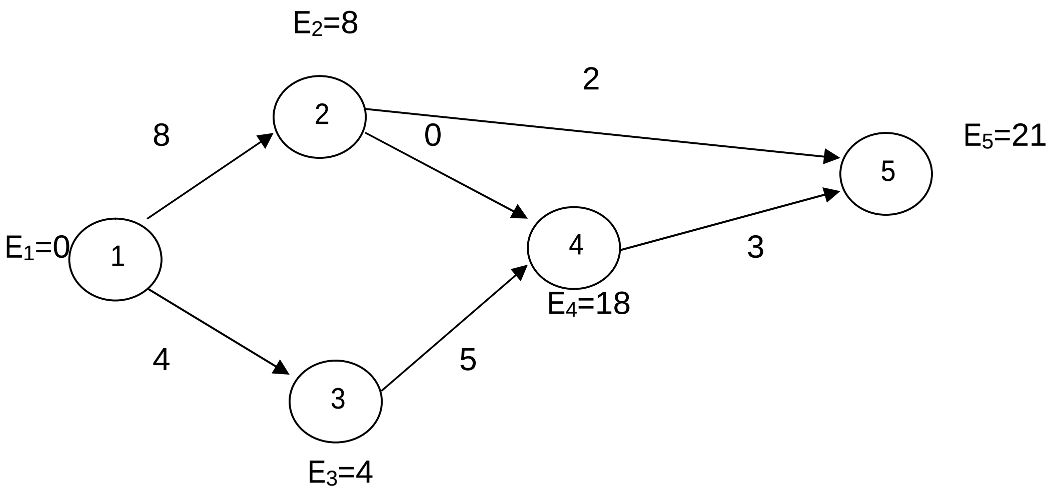
Basic Difference Between PERT and CPM	
PERT	**CPM**
It assumes *probability distribution* for the duration of each activity. Thus the completed time estimate for all the activities are needed	This technique was developed in connection with a construction and maintenance project in which duration of each activity was *known with certainty*
It emphasis on the completion of a task rather than the activities required to be performed to reach to a particular even or task, So it is also called *event oriented technique.*	It is suitable for establishing a trade - off for optimum balancing between schedule time and cost of the project.
It is used for *one time projects* involving activities of non-repetitive nature (i.e activities which may never have been performed before) in which time estimates are *uncertain*, such as redesigning an assembly line or installing a new information system.	It is used for completion of projects involving activities of *repetitive nature*.

PROBLEM:-01

Calculate earliest start and finish time of each activity of the project given below.

Activity	1-2	1-3	2-4	2-5	3-4	4-5
Duration in days	8	4	10	2	5	3

SOLUTION:-



First we have to calculate Earliest start time then Earliest finish time.

ACTIVITY i-j	Duration t_{ij}	Earliest Start $ES_{ij}=E_i= Max\{E_{i-1}+t_{ij}\}$	Earliest Finish $EF_{ij}=ES_{ij}+t_{ij}$
1-2	8	E1=0	0+8=8
1-3	4	E_1= 0	0+4=4
2-4	10	E_2= 0+8=8 [Node 2 can be reached from only one node 1]	8+10=18
2-5	2	E_2 0+8=8 [Node 2 can be reached from only one node 1]	8+2=10
3-4	5	E_3=0+4=4 [Node 3 can be reached from only one node 1]	4+5=9
4-5	3	E_4= Max{8+10, 4+5}=18 [Node 4 can be reached from node 2 and 3]	18+3=21

NOTE

(i) E_5=Max{8+2,18+3}=21, since node 5 can be reached from node 2 and node 4.

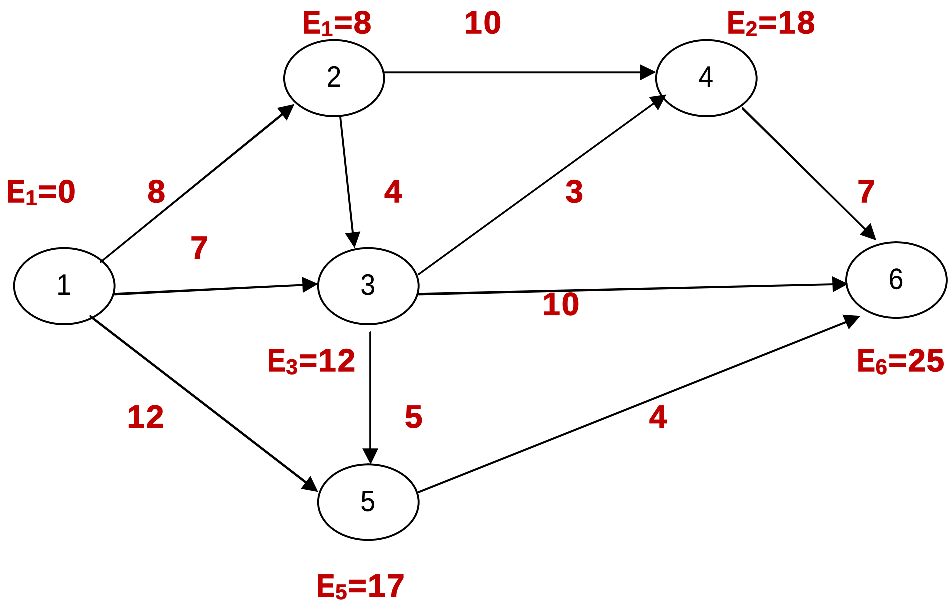
(ii) The method used here is called forward pass method, since it start from the initial node and ending with terminal node of the network diagram.

PROBLEM:-02

Calculate earliest start and finishing time of each activity of the project given below.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	5-6	4-6	3-6
Duration in week	8	7	12	4	10	3	5	4	7	10

SOLUTION:-



First we have to calculate Earliest start time then Earliest finish time.

ACTIVITY i-j	Duration t_{ij}	Earliest Start $ES_{ij}=E_i= Max\{E_{i-1}+t_{ij}\}$	Earliest Finish $EF_{ij}=ES_{ij}+t_{ij}$
1-2	8	$E_1= 0$	0+8=8
1-3	7	$E_1= 0$	0+7=7
1-5	12	$E_1= 0$	0+12=12
2-3	4	$E_2=0+8=8$ [Node 2 can be reached from only one node 1]	8+4=12
2-4	10	$E_2=0+8=8$	8+10=18
3-4	3	$E_3=max\{8+4,0+7\}=12$ [Node 3 can be reached from Node 1 ,Node 2]	12+3=15
3-5	5	$E_3=12$	12+5=17
3-6	10	$E_3=12$	12+10=22
4-6	7	$E_4= Max\{8+10,12+3\}=18$ [Node 4 can be reached from Node 2 ,Node 3]	18+7=25
5-6	4	$E_5= Max\{5+12,0+12\}=17$ [Node 5 can be reached from Node 1, Node 3]	17+4=21

NOTE:-

$E_6=max\{18+7, 12+10, 17+4\}=25$, since Node 6 can be reached from Node 3, Node 4 and Node 5.

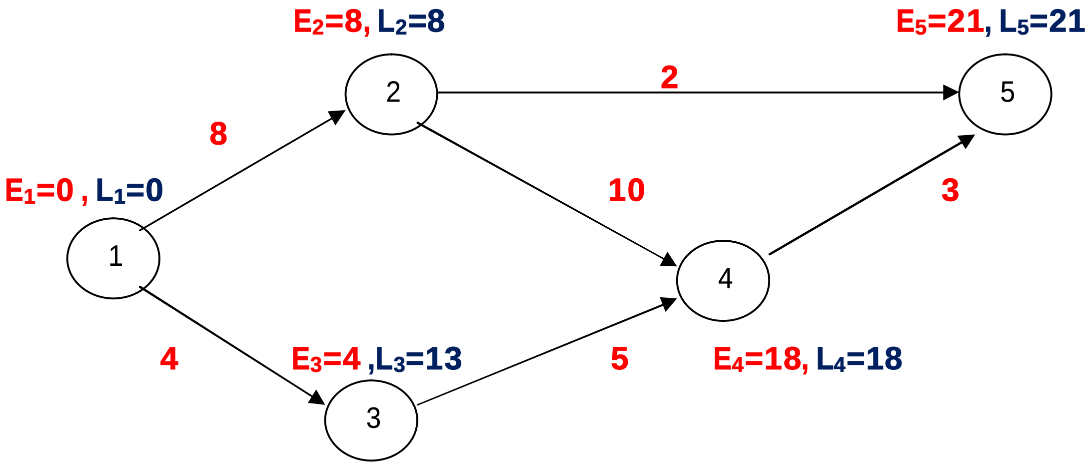
PROBLEM:-03

Calculate the latest start and finish time of the activity of the project given below.

Activity	1-2	1-3	2-4	2-5	3-4	4-5
Duration in days	8	4	10	2	5	3

SOLUTION:-

First we have to calculate earliest start and then earliest finish time.



Latest finish time has to be calculated first then latest start time.

Activity	Duration (t_{ij})	Latest Start $LS_{ij}=LF_{ij}-t_{ij}$	Latest Finish $LF_{ij}=L_j=Min\{L_{j+1}-t_{ij}\}$
1-2	8	8-8=0	L_2=Min{18-10,21-2}= 8 [Node 2 can be reached in backward direction from Node 4 and Node 5]
1-3	4	13-4=9	L_3 =13 [Node 3 can be reached in backward direction from Node 4]
2-4	10	18-10=8	L_4= 18
2-5	2	21-2=19	L_5= 21
3-4	5	18-5=13	L_4= 18 [Node 4 can be reached in backward direction from only Node 5]
4-5	3	21-3=18	L_5= 21

NOTE

(i) $L_1 = Min\{13-4, 8-8\} = 0$, since node 1 can be reached in backward direction from node 2 and node 3.

(ii) The method used here is called backward pass method, since it start from the terminal node and ending with initial node of the network diagram.

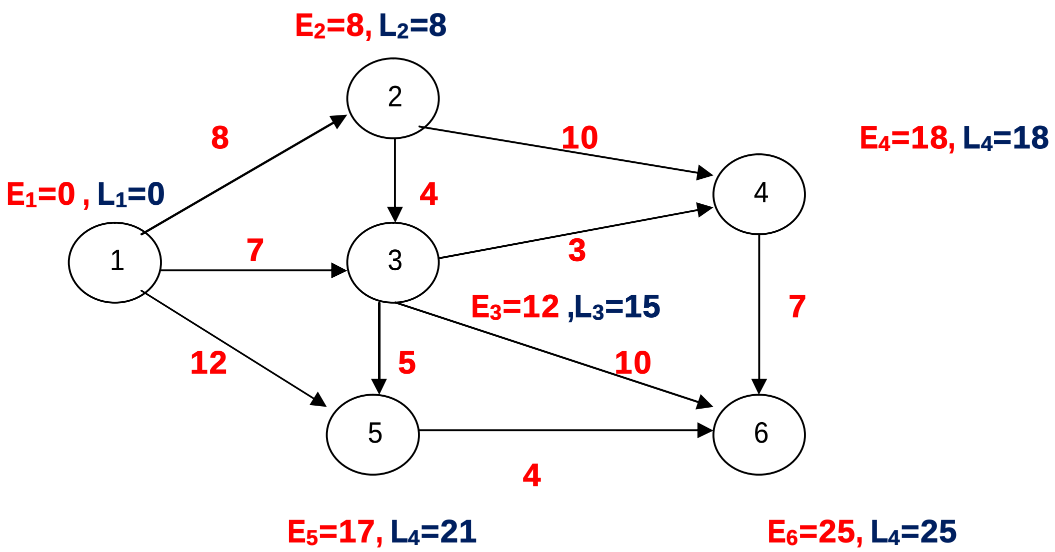
PROBLEM:-04

Calculate the latest start and finish time of the activity of the project given below.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration in week	8	7	12	4	10	3	5	10	7	4

SOLUTION:-

First we have to calculate earliest start and finish time.



Activity i-j	Duration (t_{ij})	Latest Start $LS_{ij}=LF_{ij}-t_{ij}$	Latest Finish $LF_{ij}=L_j=Min\{L_{j+1}-t_{ij}\}$
1-2	8	0	8
1-3	7	8	15
1-5	12	9	21
2-3	4	11	15
2-4	10	8	18
3-4	3	15	18
3-5	5	16	21
3-6	10	15	25
4-6	7	18	25
5-6	4	21	25

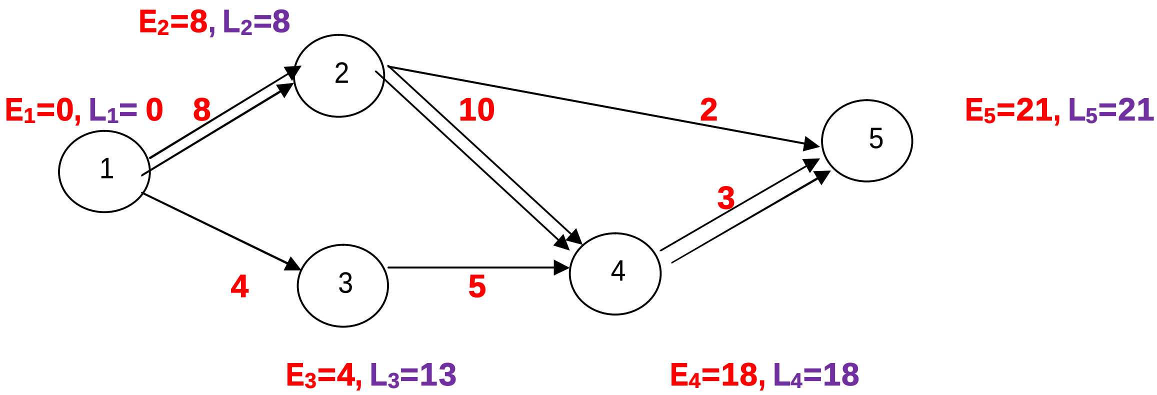
PROBLEM:-05

Calculate the latest start and finish time and earliest start and finish time of each activity of the project given below.

Activity	1-2	1-3	2-4	2-5	3-4	4-5
Duration	8	4	10	2	5	3

Also find (i) Floats (ii) find the critical path.

SOLUTION:-



Activity	**Duration**	**ES_{ij}**	**EF_{ij}**	**LS_{ij}**	**LF_{ij}**	**TF_{ij}**	**FF_{ij}**	**IF_{ij}**	**InF_{ij}**
i-j	**t_{ij}**	**E_i**	**E_i+t_{ij}**	**L_j-t_{ij}**	**L_j**	**$L_j-t_{ij}-E_i$**	**$E_j-E_i-t_{ij}$**	**$E_j-L_i-t_{ij}$**	**TF-FF**
1-2	**8**	0	8	0	8	**0**	0	0	0
1-3	4	0	4	9	13	9	0	0	0
2-4	**10**	8	18	8	18	0	0	0	0
2-5	2	8	10	19	21	11	11	11	0
3-4	5	4	9	13	18	9	9	0	0
4-5	**3**	18	21	18	21	**0**	0	0	0

The critical path is the above network is 1-2-4-5 and the longest duration is 21.

NOTE:-

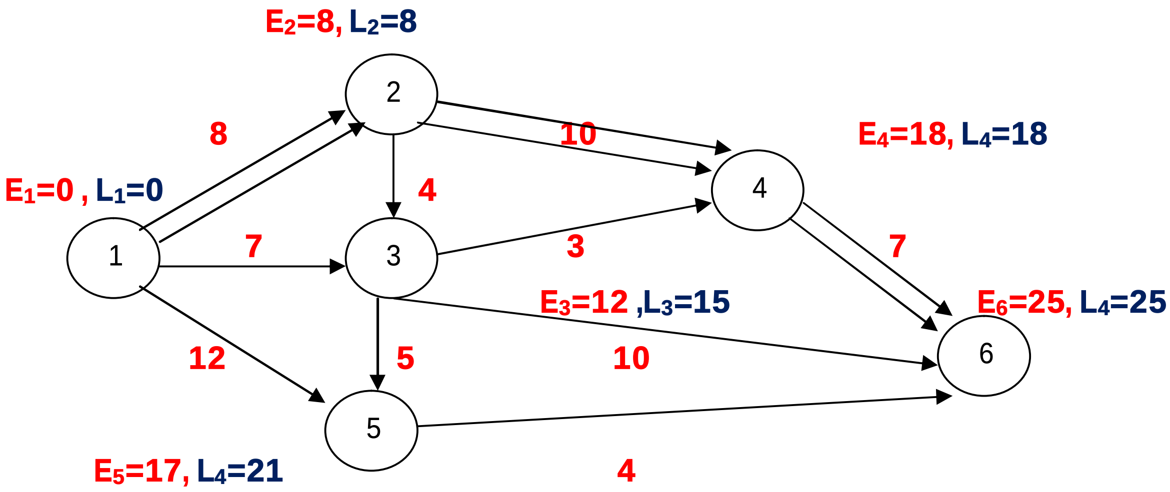
If total float is zero, then the corresponding activity is called *critical activity*.

PROBLEM:-06

Find (i) critical path (ii) Earliest start , finish, (iii) Latest start, finish (iv) value of float for the following.

Activity	1-2	1-3	1-5	2-3	2-4	3-4	3-5	3-6	4-6	5-6
Duration	8	7	12	4	10	3	5	10	7	4

SOLUTION :-



Activity i-j	Duration t_{ij}	ES_{ij} E_i	EF_{ij} E_i+t_{ij}	LS_{ij} L_j-t_{ij}	LF_{ij} L_j	TF_{ij} $L_j-t_{ij}-E_i$	FF_{ij} $E_j-E_i-t_{ij}$	IF_{ij} $E_j-L_i-t_{ij}$	InF_{ij} TF-FF
1-2	8	0	8	0	8	0	0	0	0
1-3	7	0	7	8	15	8	5	5	3
1-5	12	0	12	9	21	9	5	5	4
2-3	4	8	12	11	15	3	0	0	3
2-4	10	8	18	8	18	0	0	0	0
3-4	3	12	15	15	18	3	3	0	0
3-5	5	12	17	16	21	4	0	-3	4
3-6	10	12	22	15	25	3	3	0	0
4-6	7	18	25	18	25	0	0	0	0
5-6	4	17	21	21	25	4	4	0	0

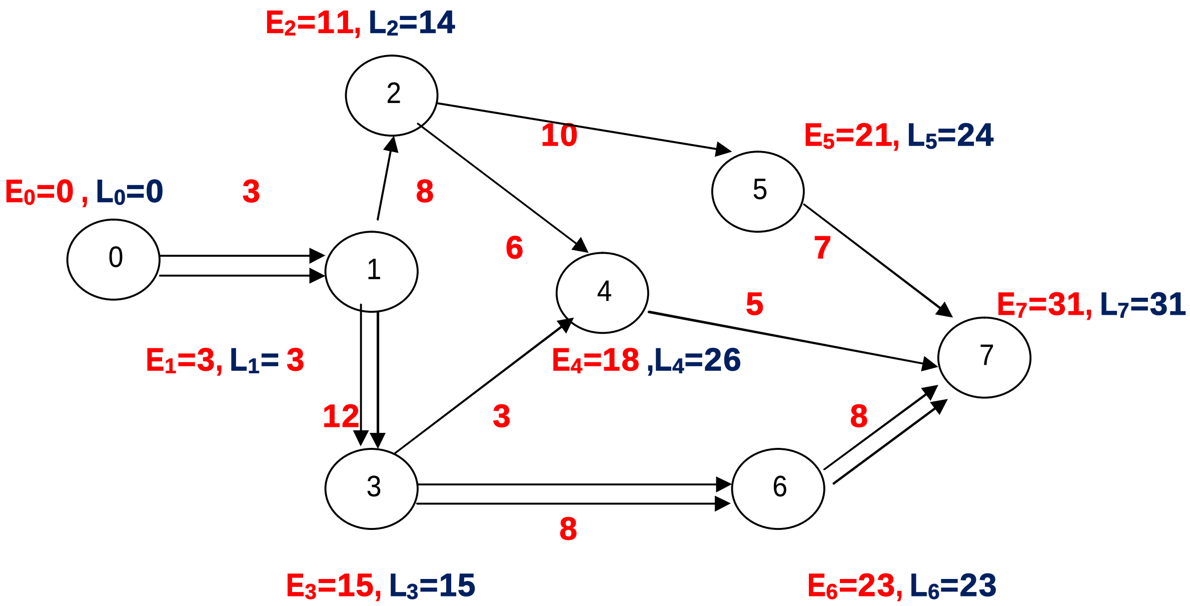
The critical path is 1-2-4-6 and project completion time 25.

PROBLEM:-07

Construct the network for the project whose activity are given below and find the critical path and its project duration and also float values.

Activity	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration	3	8	12	6	3	3	8	5	3	8

SOLUTION:-



Activity i-j	Duration t_{ij}	ES_{ij} E_i	EF_{ij} E_i+t_{ij}	LS_{ij} L_j-t_{ij}	LF_{ij} L_j	TF_{ij} $L_j-t_{ij}-E_i$	FF_{ij} $E_j-E_i-t_{ij}$	IF_{ij} $E_j-L_i-t_{ij}$	InF_{ij} $TF_{ij}-FF_{ij}$
0-1	**3**	0	3	0	3	0	0	0	0
1-2	8	3	11	3	11	0	0	0	0
1-3	**12**	3	15	3	15	0	0	0	0
2-4	6	11	17	20	26	9	1	-2	8
2-5	3	11	14	25	26	12	0	-3	12
3-4	3	15	18	23	28	10	0	0	10
3-6	**8**	15	23	15	26	3	0	0	3
4-7	5	18	23	26	23	0	8	0	-8

5-7	3	14	17	28	31	14	3	0	11
6-7	**8**	23	31	23	31	0	0	0	0

The critical path is 0-1-3-6-7 and the project duration is 31.

PROBLEM-08

The following table indicates the details of the project. The duration are in days 'a' refers to optimistic time, 'm' refer to most taking time, 'b' refer to estimate time duration.

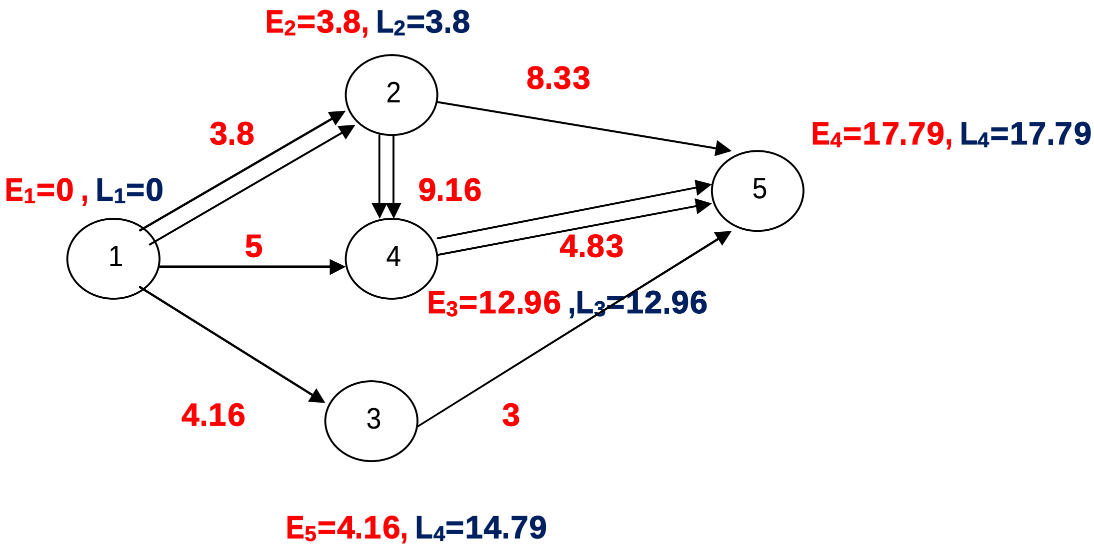
Activity	$t_o(a)$	$t_m(m)$	$t_p(b)$
1-2	2	4	5
1-3	3	4	6
1-4	4	5	6
2-4	8	9	11
2-5	6	8	12
3-5	2	3	4
4-5	2	5	7

(i) Draw the network

(ii) Find the critical path

(iii) Determine the expected S.D of the completion time.

SOLUTION:-



Activity	$t_o(a)$	$t_m(m)$	$t_p(b)$	$t_e = \frac{1}{6}(t_0 + 4t_m + t_p)$	$\sigma^2 = \left[\frac{1}{6}(t_p - t_0)\right]^2$
1-2	2	4	5	**3.8**	**0.5**
1-3	3	4	6	4.16	0.5
1-4	4	5	6	5	0.3
2-4	8	9	11	**9.16**	**0.5**
2-5	6	8	12	8.33	1
3-5	2	3	4	3	0.3
4-5	2	5	7	**4.83**	**0.33**

Therefore the Critical path is *1-2-4-5* and its total duration is *17.79(=3.8+9.16+4.83)*

The expected standard deviation of critical path is 1.33 (=10.5+0.5+0.33)

PROBLEM:-09

A project consist of the following activities and tome estimates t_o-least time in day , t_p-greatest time in days

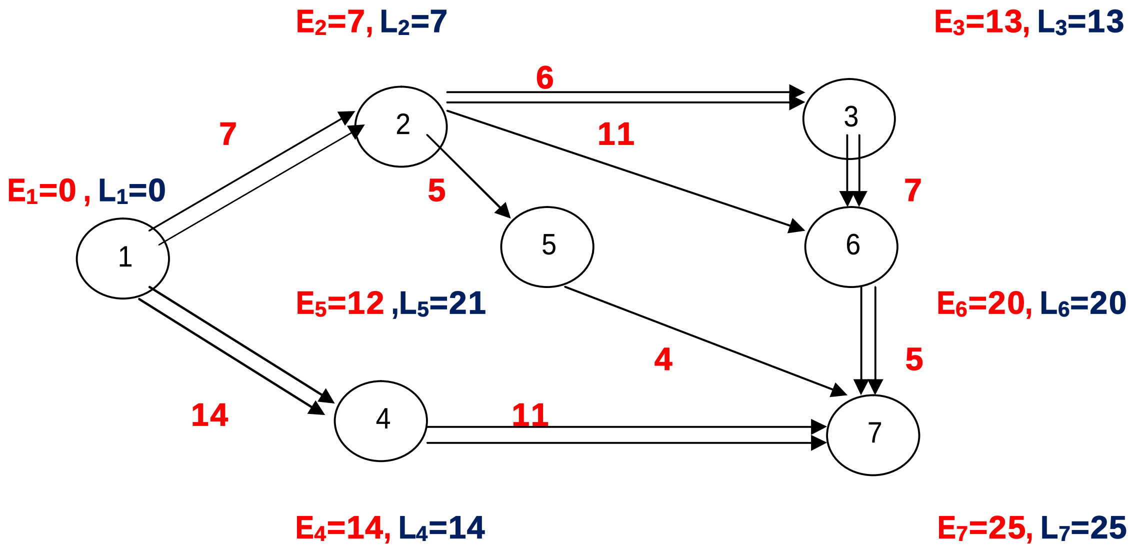
Activity	t_o	t_p	t_m
1-2	3	15	6
2-3	2	14	5
1-4	6	30	12
2-5	2	8	5
2-6	5	17	11
3-6	3	15	6
4-7	3	27	9
5-7	1	7	4
6-7	2	8	5

(i) Prove the network

(ii) What is the probability that project will be completed in 27 days

SOLUTION:-

Activity	t_o	t_p	t_m	t_e	SD	σ^2
1-2	3	15	6	**7**	**2**	**4**
2-3	2	14	5	**6**	**2**	**4**
1-4	6	30	12	**14**	**4**	**16**
2-5	2	8	5	5	1	1
2-6	5	17	11	11	2	4
3-6	3	15	6	**7**	**2**	**4**
4-7	3	27	9	**11**	**4**	**16**
5-7	1	7	4	4	1	1
6-7	2	8	5	**5**	**1**	**1**



Critical paths are

(i) 1-2-3-6-7 and its duration 25 (=7+6+7+5)

(ii) 1-4-7 and its duration 25 (=14+11)

Variance are

(i) Critical path 1-2-3-6-7 variance is 13 (=4+4+4+1)

(ii) 1-4-7 is 16+16=32

Let t_S- denotes the project completion time

P(project will be completed in 27 days)= $p(t_s \leq 27)$

$$= p(t_s - t_e \leq 27-25)$$

$$= p\left(\frac{t_s - t_e}{\sigma} \leq \frac{27-25}{5.65}\right)$$

$$= p(Z \leq 0.35)$$

$$= p[-\infty < Z \leq 0] + P[0 < Z < 0.35]$$

=0.5+0.1368

=0.6368

P(project will be completed in 27 days)=63.7%

PROBLEM:-10

The time estimate in month of all the activities of the project are given below.

Activity	A	m	b
1-2	0.8	1.0	1.2
2-3	3.7	5.6	9.9
2-4	6.2	6.6	15.4
3-4	2.1	2.7	6.1
4-5	0.8	3.4	3.6
5-6	0.9	1.0	1.1

i) find the expected duration and S.D to each activity.

ii) construct the project network.

iii)Determine the critical path, expected length and expected variance of the project length.

iv)What is the probability that the project will be completed.

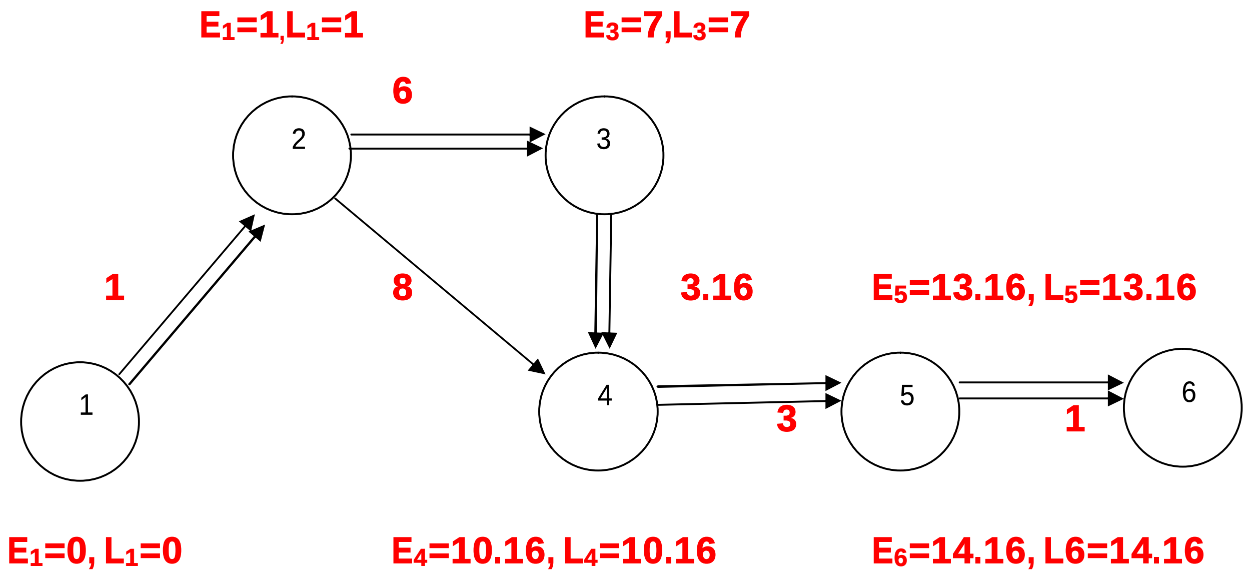
a) two month later the expected

b)three month before the expected

c)what new day has above 90% chances of being met?

SOLUTION:-

Activity	A	m	b	t_e	SD	σ^2
1-2	0.8	1.0	1.2	**1**	**0.067**	**0.0048**
2-3	3.7	5.6	9.9	**6**	**1.033**	**1.0677**
2-4	6.2	6.6	15.4	8	1.533	2.3511
3-4	2.1	2.7	6.1	**3.16**	**0.667**	**0.4448**
4-5	0.8	3.4	3.6	**3**	**0.467**	**0.2170**
5-6	0.9	1.0	1.1	**1**	**0.333**	**0.001**



Critical path 1-2-3-4-5-6 and expected length of the critical path is 14.16

Expected variance of the critical path1-2-3-4-5-6 is 1.733

(=0.004+1.067+0.444+0.217+0.001)

SD=$(1.733)^{1/2}$=1.31

(iv)

(a)

Let t_S be the project completion time

P(Project will be completed two month later)=P(t_S<= 14.16+2)

$$= P(t_s \leq 16.16)$$

$$= P(t_s - t_e \leq 16.16 - 14.16)$$

$$= P\left(\frac{t_s - t_e}{\sigma} \leq \frac{2}{1.31}\right)$$

$$= p(Z \leq 1.53)$$

$$= p[-\infty < Z \leq 0] + P[0 < Z \leq 1.53]$$

=0.5+0.4370

=0.937=93.7%

(iv)

(b)

P(Project will be completed before three month than expected)

$$= P(t_s \leq 11.16)$$

$$= P\left(\frac{t_s - t_e}{\sigma} \leq \frac{11.16 - 14.16}{1.31}\right)$$

$$= P(Z \leq -2.29)$$

$$= P[-\infty < Z \leq -2.30] = P[2.30 \leq Z < \infty]$$

$$= P[0 \leq Z < \infty] - P[0 \leq Z \leq 2.30]$$

=0.5-0.4893

=0.0107

=1.07%

(iv)

(c)

σ =Standard diviation of the project length

t_s= due date, schedule date

t_e=expected duration of the project length

$$z = \frac{t_s - t_e}{\sigma} = \frac{t_s - 14.16}{1.31}$$

Given:

$$p\left(Z \leq \frac{t_s - 14.16}{1.31}\right) = 90\% = 0.9$$

$$p[-\infty < Z < 0] + P[0 < Z \leq \frac{t_0 - 14.16}{1.31}] = 0.9$$

$$P[0 \leq Z \leq \frac{t_s - 14.16}{1.31}] = 0.4$$

$$P[0 \leq Z \leq \frac{t_s - 14.16}{1.31}] = P[0 \leq Z \leq 1.287]$$

$$\left(\frac{t_s - 14.16}{1.303}\right) = 1.28$$

T_s=15.82 month nearly.

UNIT-V

Non-linear programming method(NLPP or NLP)

If either objective function or constraints are non linear, then it is called non - linear programming problem. it in short written by NLP or NLPP.

Note:-

Linearity means the variables should be of degree one and their are not multiplied together.

Local minimum / Relative minimum

A function of one variable f(x) is said to have a local minimum at $x=x_0$, if $f''(x_0)>0$

Local maxima / Relative maxima

A function of one various f(x) is said to have local maxima at $x=x_0$, if $f''(x_0)<0$

Note:-

If $f''(x)=0$, neither maximum nor minimum exist for f(x).

Maxima and minima using higher order derivatives.

When the function consist of more than one variable, we use the hessian matrix formula.

$$H(x_0) = \left(\frac{\partial^2 f}{\partial x_i \partial y_j} \right)_{x=x_0}$$

To decide the nature of function f, we use the following

(I) f(x) is relative minimum at $x=x_0$,if $H(x_0)$ is positive definite.

(II) f(x) is relative maximum at $x=x_0$, if $H(x_0)$ is negative definite.

(III) There exist a point of inflexion at $x=x_0$ (saddle point), if $H(x_0)$ is indefinite.

Necessary and sufficient condition for the existence of local maxima and minima and point of inflexion.

Necessary condition	Sufficient condition	Nature of function	Conclusion
$f'(x_0)=0$	$f''(x_0)=\dots=f^{(n-1)}(x_0)=0$ and $f^{(n)}(x_0)<0$, n even	Concave	Local maximum at $x=x_0$
$f'(x_0)=0$	$f''(x_0)=\dots=f^{(n-1)}(x_0)=0$ and $f^{(n)}(x_0)>0$, n even	Convex	Local minimum at $x=x_0$
$f'(x_0)=0$	$f''(x_0)=\dots=f^{(n-1)}(x_0)=0$ and $f^{(n)}(x_0)\neq 0$, n even	-	Point of inflexion at $x=x_0$

Formation of non-linear programming problem(NLP)

Problem:-01

An engineering company has received a rush order for a maximum number of two types of items that can be produced and transported during a two weeks period. The profit in 100rs on this order is related to number of each type of item manufactured by the company and is given by $12x_1+10x_2-x_1^2-x_2^2+61$ where x_1 is the number of units in 1000 of type I, x_2 is the number of units in 1000 of type II item . because of other commitments over the next two weeks. The company has available only 60 hours in the shifting and packing department if is estimated that every 1000 units of the type I and type II item will require 20hrs and 30hrs respectively in the shifting and packing department. Given the above information, how many units of each type of item should the company produce, in order to maximum the profit.

Solution:-

Here x_1 is the number of units in 1000 of type I

and x_2 is the number of units in 1000 if type II

The profit is given by $12x_1+10x_2-x_1^2-x_2^2+61$

i.e Max $Z= 12x_1+10x_2-x_1^2-x_2^2+61$

subject to constraints,

Since shifting and packing a unit of item of type I require 20hr

Total time required for shifting and packing of type I item = $20x_1$

Since shifting and packing a unit of item of type I require 20hr

Total time required for shifting and packing of type II item= $30x_2$

Total time required for shifting and packing of type I and type II item=$20x_1+30x_2$

Maximum available time for shifting and packing type I and type II items =60hr

$20x_1+30x_2\leq 60$

$x_1,x_2\geq 0$

In this problem object function is non-linear and constraints are linear

Therefore the given problem is non- linear programming problem.

Problem:-02

The company sells two types of items A and B. Item A sells for rs.25 per unit. No quantity discount is given. The sales revenue for item B, decreases as the number of its unit sold increases and is given by,m$30x_2-0.3x_2^2$, where $x_2$nis the number of unit sold of item B.The marketing department has only 1200 hrs available for distributing this items in the next year, further the company estimates the sales time function is non linear and is given by, $x_1+0.2x_1^2+3x_2+0.35x_2^2$, the company can produce 6000 units of item A and B for sales in the next year, given the above information, how many unit of each type of item should the company produce.

Solution:-

Here x_1 is the number of units sold of type A

and x_2 is the number of units sold of type B.

Total profit= sales revenue of item A+ sales revenue of item B

Max Z= sales revenue of item A+ sales revenue of item B

Since Item A sells for rs.25 per unit

Sales revenue of item A = $25x_1$

Sales revenue of item B=$30x_2-0.3x_2{}^2$

Max Z=$25x_1+30x_2-0.3x_2^2$

Since the marketing department has only 1200 hrs for distributing this item next year.

Sale time function is given by $x_1+0.2x_1^2+3x_2+0.35x_2^2$

$x_1+0.2x_1^2+3x_2+0.35x_2^2 \le 1200$

The company produce 6000 units of item A and B.

x_1+x_2=number of item A and B sold

i.e. $x_1+x_2 \le 6000$

and $x_1,x_2 \ge 0$

Since both objective function and conditions are non linear, therefore It is a NLP.

SOLUTION OF NLP BY GRAPHICAL METHOD

Problem:=01

Solve the following NLP using graphical method

Max $Z=2x_1+3x_2$

Sub to constraints

$x_1^2+x_2^2 \le 20$,

$x_1.x_2 \le 8$

and $x_1,x_2 \ge 0$

Solution:-

Let us plot the constraints in graph

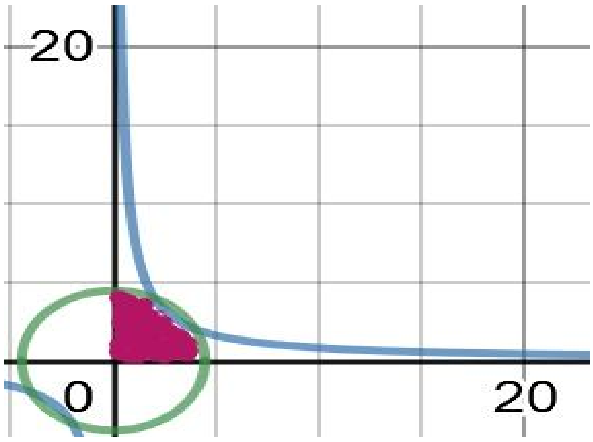
First constraints becomes, $x_1^2+x_2^2= 20$

i.e we have to draw the circle with centre (0,0) radius $\sqrt{20}$

The second constraints becomes, $x_1.x_2=8$

i.e we have to draw the rectangular hyperbola

The shaded region is called feasible region



To find the intersection point

solving $x_1^2+x_2^2= 20$ and $x_1.x_2=8$

Substitute $x_1 = \frac{8}{x_2}$, we get

$$\frac{64}{x_2^{\ 2}} + x_2^{\ 2} = 20$$

$64+x_2^4= 20x_2^2$

$x_2^4- 20x_2^2-64=0$

$(x_2^2)^2-20x_2^2+64=0$

$w^2-20w+64=0$, $w=x^2_2$

$$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$w = \frac{20 \pm \sqrt{200 - 256}}{2}$$

$$w = \frac{20 \pm \sqrt{144}}{2}$$

$w = 10 \pm 26$

$w=16,4$

$x_2^2=16,4$

$x_2=4,-4,2,-2$ (negative values should be avoided, since x_1,x_2 are nonnegative)

Sub $x_2=4,2$ in $x_1x_2=8$, we get

$x_1=2,4$

i.e the intersection points are (2,4)and (4,2)

Let us find x_1,x_2 within the region OABCD when the value of objective function is maximum .

Let us locate the point (x_1,x_2) using ISO profit function approach

ISO- profit method

Let $2x_1+3x_2=k$

Let k=0

Draw $2x_1+3x_2=0$

Let k=6

Draw $2x_1+3x_2=6$

Let k=12

Draw $2x_1+3x_2=12$

Let k=14

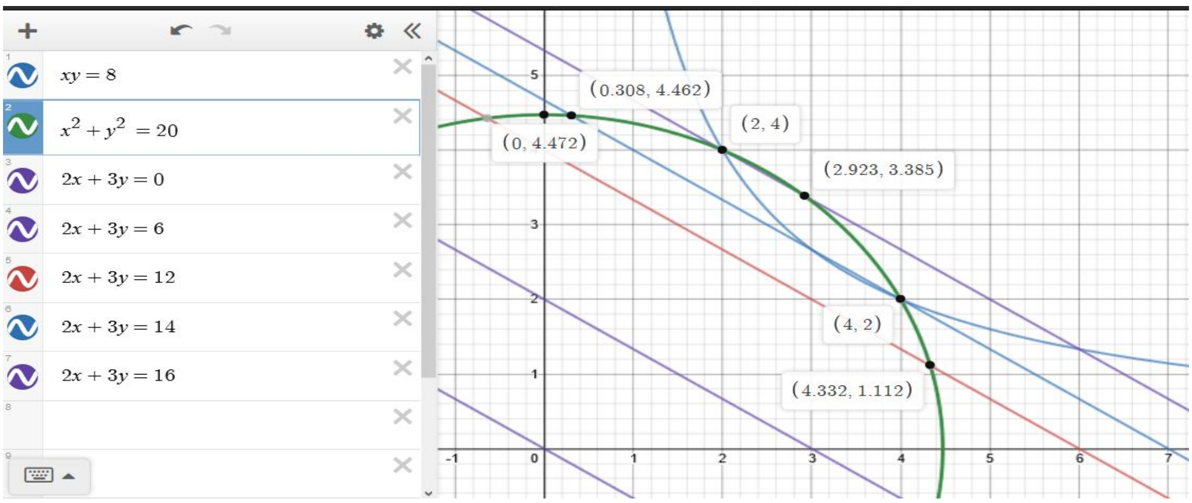
Draw $2x_1+3x_2=14$

Let k=16

Draw $2x_1+3x_2=16$

Draw parallel objective function lines for different constant values of K.

Starting with 0 and so on we find that the ISO profit line with k=16 such as the extreme boundary points c(2,4) where the value of Z is maximum.



Hence the optimum solution is $x_1=2$, $x_2=4$.

Note:-

In the above problem the objective function line 2x+3y=14, intersecting with point (4,2), but the next objective function line 2x+3y=16 , intersecting with point (2,4). Among the points (2,4) gives the maximum for objective function.

Problem:-02

Solve graphically the following NLP

Max $Z=8x_1-x_1^2+8x_2-x_2^2$

Subject to constraints

$x_1+x_2\leq 12$

$x_1-x_2\geq 4$ and

$x_1,x_2\geq 0$

Solution:-

Let us plot the constraints in graph

Let us draw the first constraint $x_1+x_2= 12$

$x_1=0 \quad x_2=12$

$x_2=0 \quad x_1=12$

It is a line joining two points(0,12) &(12,0)

Let us draw the second constraint $x_1-x_2=4$

$x_1=0 \quad x_2=-4$

$x_2=0 \quad x_1=4$

It is the line joining two points (0,-4)&(4,0)

To find the intersection points

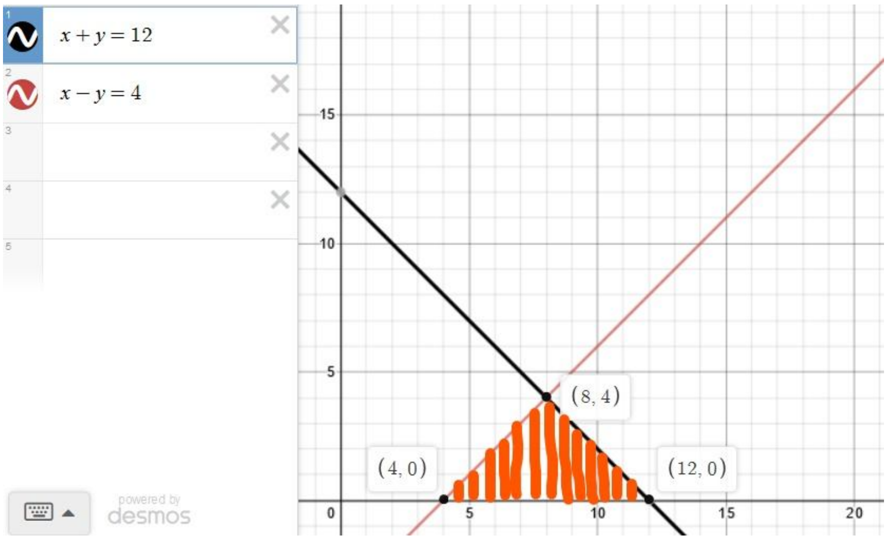
$x_1+x_2= 12$, $x_1-x_2=4$

Add the above two equations, we have $2x_1 \quad =16$

$=>x_1=8$

Substitute the value of x_1, we have $x_2=4$

Therefore the intersection point is (8,4)



The shaded region is the feasible region.

Let us plot the objective function

$8x_1-x_1^2+8x_2-x_2^2=0$

$2.4x_1-x_1^2+2.4x_2-x_2^2=0$

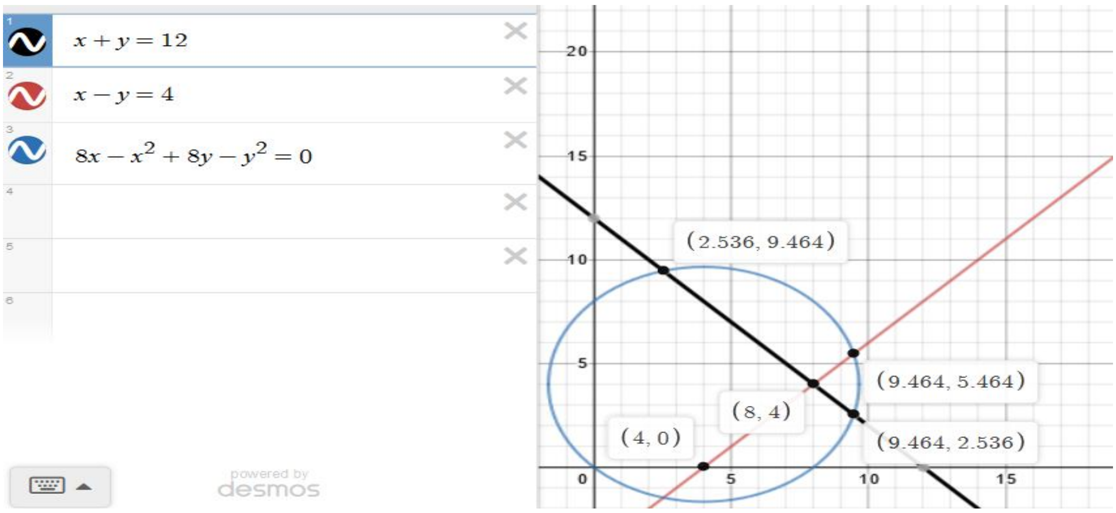
$-(x_1^2-2.4x_1+4^2-4^2)-(x_2^2-2.4x_2+4^2-4^2)=0$

$-((x_1-4)^2-16)-((x_2-4)^2-16)=0$

$-(x_1-4)^2+16-(x_2-4)^2+16=0$

$(x_1-4)^2+(x_2-4)^2=32$

It is a circle with centre (4,4) and radius $(32)^{1/2}$



The feasible region is the convex region, the optimum point (x_1, x_2) must be somewhere in the convex region A,B,C also it would be the at which the side of the convex region with tangent to the circle

$Z=8x_1-x_1^2+8x_2-x_2^2$

To find the optimum point (x_1, x_2)

We use gradient method

The gradient of the tangent to the circle can be obtained by differentiating the equation w.r.t x_1 and equating to zero

$8x_1-x_1^2+8x_2-x_2^2=0$

$$8-2x_1+8\frac{dx_2}{dx_1}-\frac{dx_2{}^2}{dx_1}=0$$

$$8-2x_1+8\frac{dx_2}{dx_1}-\frac{dx_2{}^2}{dx_2}.\frac{dx_2}{dx_1}=0$$

$$8-2x_1+8\frac{dx_2}{dx_1}-\frac{dx_2{}^2}{dx_2}.\frac{dx_2}{dx_1}=0$$

$$8-2x_1+8\frac{dx_2}{dx_1}-2x_2.\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1}=\frac{2x_1-8}{8-2x_2}\text{\_\_\_\_\_III}$$

Gradient of the line

$x_1+x_2= 12$

$1+\frac{dx_2}{dx_1}=0$

$\frac{dx_2}{dx_1}=-1$--------IV

Gradient of the line

$x_1-x_2=4$

$1-\frac{dx_2}{dx_1}=0$

$\frac{dx_2}{dx_1}=1$____V

Sub (IV) in (III)

$$-1=\frac{2x_1-8}{8-2x_2}$$

$-8+2x_2-2x_1+8=0$

$2x_2-2x_1=0$

$x_2-x_1=0$---------VI

$x_1=x_2$

Solve (VI) and (I)

$x_1+x_2= 12$

$x_2-x_1=0$

$2x_2=12$

$x_2=6$

$x_1=6$

The intersection point is (6,6)

it does not lies in feasible region, therefore it is not a optimum point

Sub (V)&(III)

$$1=\frac{2x_1-8}{8-2x_2}$$

$8-2x_2-2x_1+8=0$

$2x_2+2x_1-16=0$

$x_1+x_2=8$ -----VII

solve (VII)&(II)

$x_1+x_2=8$

$x_1-x_2=4$

$2x_1=12$

$x_1=6$

$x_2=2$

The intersection point is (6,2)

It lies in the feasible region and satisfy both the constraints

Hence the optimum solution is $x_1=6,x_2=2$ and Max Z= 24

Problem :-3

Solve the following NLP problem.

Min Z = $x_1^2+x_2^2$

Subject to

$x_1+x_2\geq 8$

$x_1+2x_2\geq 10$

$2x_1+x_2\geq 10$

$x_1,x_2\geq 0$

Solution:-

Let us plot the constraints in graph

Let us draw the first constraint $x_1+x_2= 8$____(1)

$x_1=0$ $x_2=8$

$x_2=0$ $x_1=8$

It is the line joining the points (0,8)&(8,0)

Let us draw the second constraint $x_1+2x_2= 10$___(2)

$x_1=0$ $x_2=5$

$x_2=0$ $x_1=10$

It is the line joining the points(0,5)&(10,0)

Let us draw the third constraint $2x_1+x_2= 10$_____(3)

$x_1=0$ $x_2=10$

$x_2=0$ $x_1=5$

It is the line joining the points (5,0)&(0,10)

The required region is unbounded convex region

To find the intersection points

Solve (2)&(3)

$2x_1+4x_2= 20$

$2x_1+x_2= 10$

$3x_2=10$

$x_2=3.3$

therefore $x_1=3.4$

The intersection points is E(3.4,3.3)

Solve (3)&(1)

$2x_1+x_2= 10$

$x_1+x_2= 8$

$x_1=2$

$x_2=6$

The intersection point is C(2,6)

Solve (1)&(2)

$x_1+x_2= 8$

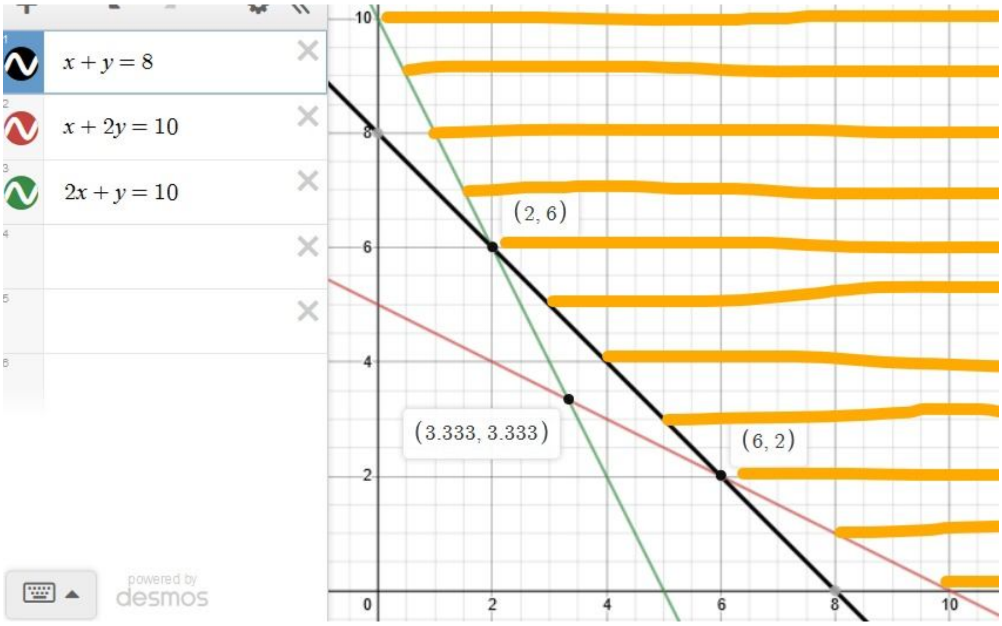
$x_1+2x_2= 10$

$-x_2=-2$

$x_2=2$

$x_1=6$

The intersection point is B (6,2)



The shaded region is feasible region.

Let us draw the objective function in graph

Let $x_1^2+x_2^2=k$

The objective function is the circle with centre (0,0)

The optimum point (x_1, x_2) must lie in the feasible convex region also it would be at which the side of the convex region with tangent to the $x_1^2+x_2^2=k$

To find optimum point

We use gradient method

The gradient of the tangent to the circle can be obtained by differentiating the equation w.r.t x_1 and equating to zero

$x_1^2+x_2^2=0$

$$2x_1+\frac{dx_2{}^2}{dx_1}=0$$

$$2x_1+\frac{dx_2{}^2}{dx_2}.\frac{dx_2}{dx_1}=0$$

$$2x_1+2x_2\frac{dx_2}{dx_2}.\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2}$$ _______(4)

Gradient line

$x_1+x_2= 8$

$$1+\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1}=-1$$ -----(5)

Gradient line

$x_1+2x_2= 10$

$$1+2\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1}=-\frac{1}{2}$$ -----(6)

Gradient line

$2x_1+x_2= 10$

$$2+\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1}=-2$$ ------(7)

Sub (5) in (4)

$$-1=\frac{-x_1}{x_2}$$

$-x_2=-x_1$

$x_1-x_2=0$-------(8)

solve (8)&(1)

$x_1-x_2=0$

$x_1+x_2=8$

____

$2\,x_1=8$

$x_1=4$

$x_2=4$

The intersection point is (4,4)

Sub (6) in (4)

$$-\frac{1}{2}=-\frac{x_1}{x_2}$$

$-x_2=-2x_1$

$-x_2+2x_1=0$---------(9)

Solve(9)&(2)

$2x_1-x_2=0$

$2x_1+4x_2= 20$

$5\ x_2=20$

$x_2=4$

$x_1=2$

The intersection point is (2,4)

Sub (7) in (4)

$$-2=\frac{-x_1}{x_2}$$

$-2x_2+x_1=0$--------(10)

Solve (10)&(3)

$-4x_2+2x_1=0$

$2x_1+x_2= 10$

$-5\ x_2=10$

$x_2=2$

$x_1=4$

The intersection point is (4,2)

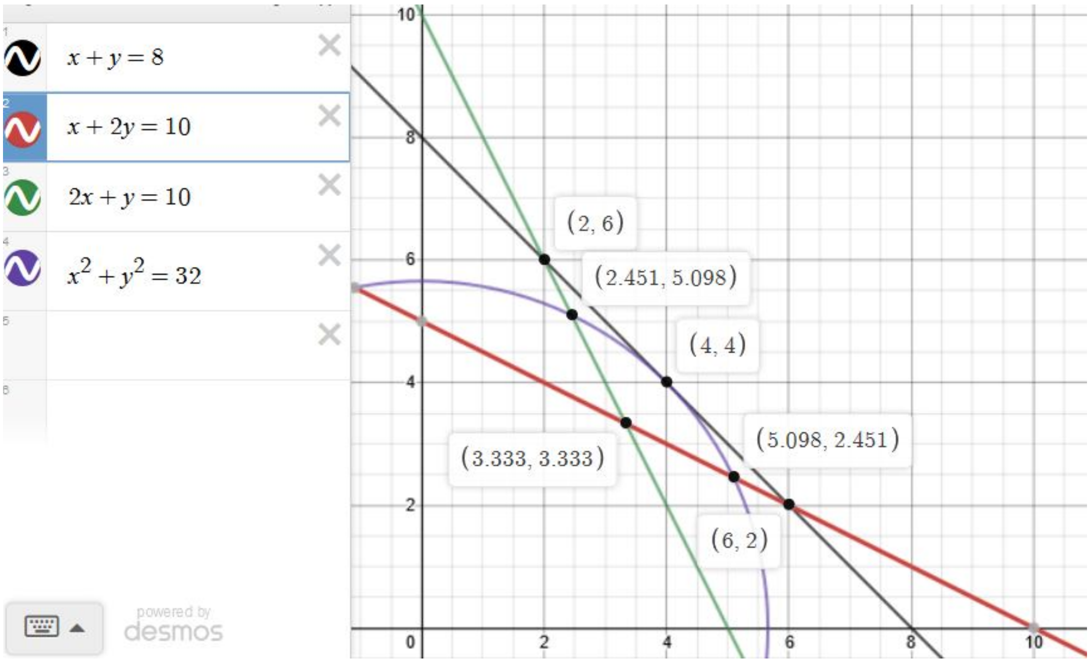
Since the (2,4),(4,2)does not lie in the feasible region and (4,4) lie in the feasible region

Therefore the optimum solution is $x_1=4$, $x_1=4$ and Max Z =32

Note:-

In the above problem the final optimum solution is at (4,4) and $z=x_1^2+x_2^2=32$.

If we plot this objective function, we get the clear picture of the optimum solution.



Problem:-04

Solve the following NLP

min Z $=10x_1- x_1^2+10x_2-x_2^2$

subject to

$x_1+x_2\leq 12$

$x_1-x_2\leq 6$

$x_1,x_2\geq 0$

Solution:

Let us plot the constraints in graph

Let us draw first constraint $x_1+x_2= 12$------(1)

$x_1=0$ $x_2=12$

$x_2=0$ $x_1=12$

It is the line joining the points(0,12)&(12,0)

Let us draw the second constraints $x_1-x_2= 6$------(2)

$x_1=0$ $x_2=-6$

$x_2=0 \quad x_1=6$

It is a line joining the points(0,-6)&(6,0)

To find the intersection points

Solve (1)&(2)

$x_1+x_2= 12$

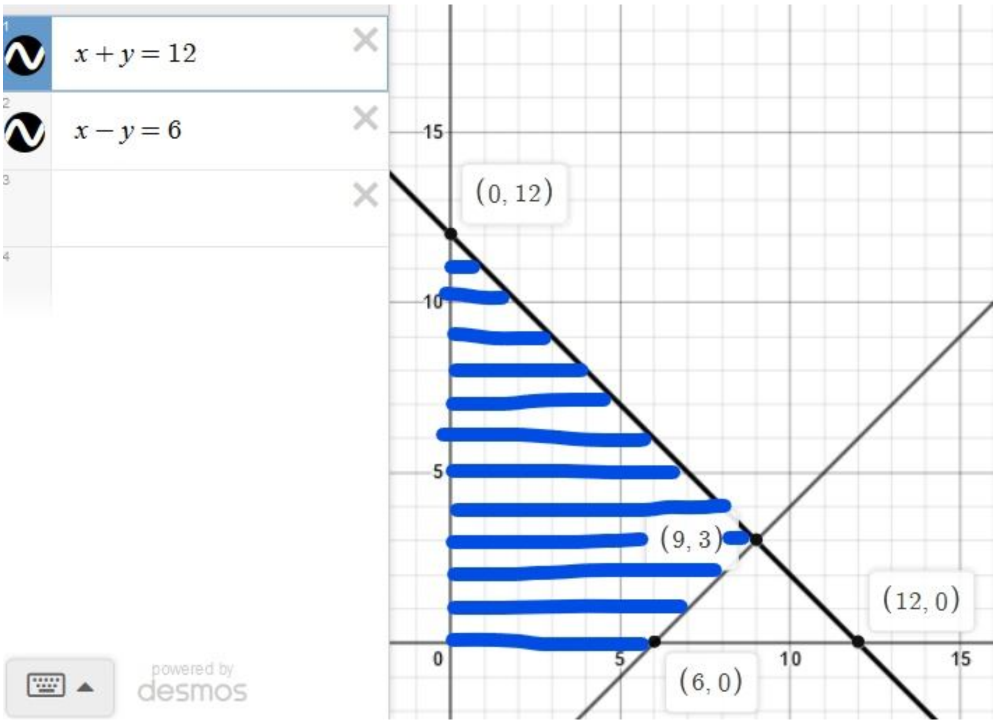
$x_1-x_2= 6$

$2\ x_2=6$

$x_2=3$

$x_1=9$

The intersection point is C(9,3)



Let us draw the objective function

let $10x_1- x_1^2+10x_2-x_2^2=0$

$2.5x_1- x_1^2+2.5x_2-x_2^2=0$

$-(x_1^2-2.5x_1+5^2-5^2)-(x_2^2-2.5x_2+5^2-5^2)=0$

$-((x_1-5)^2-25)-((x_2-5)^2-25)=0$

$(x_1-5)^2-25+(x_2-5)^2-25=0$

$(x_1-5)^2+(x_2-5)^2=50$

It is a circle with centre (5,5) radius 50

The feasible region is the convex region the optimum point(x_1,x_2) must be somewhere in the convex region A,B,C also it would be at which circle side of the convex region with tangent to the circle $10x_1- x_1^2+10x_2-x_2^2=k$.

To find optimum point

We use gradient method

The gradient of the tangent to the circle can be obtained by differentiating the equation w.r.t x_1 and equating to zero

$10x_1- x_1^2+10x_2-x_2^2=0$

$$10-2x_1+10\frac{dx_2}{dx_1}-\frac{dx_2}{dx_1}^2=0$$

$$10-2x_1+\frac{dx_2}{dx_1}-2x_2\frac{dx_2}{dx_1}^2=0$$

$$\frac{dx_2}{dx_1}=\frac{2x_1-10}{10-2x_2}\text{-------(3)}$$

Gradient of line

$x_1+x_2= 12$

$$1+\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1}=-1\text{-----(4)}$$

Gradient of line

$x_1-x_2= 6$

$$1-\frac{dx_2}{dx_1}=0$$

$$\frac{dx_2}{dx_1}=1\text{-------(5)}$$

Sub (4)in (3)

$$-1=\frac{2x_1-10}{10-2x_2}$$

$-10-2x_2-2x_1+10=0$

$2x_1-2x_2=0$

x_1-x_2=0-------(6)

solve (6)&(1)

$x_1+x_2= 12$

$x_1-x_2=0$

$2x_1=12$

$x_1=6$

$x_2=6$

The intersection point is (6,6)

It is not in the feasible region, therefore it is not a optimum point

Sub(5) in (3)

$$1 = \frac{2x_1 - 10}{10 - 2x_2}$$

$10-2x_2-2x_1+10=0$

$2x_2+2x_1-20=0$

$x_1+x_2=10$-----(7)

Solve (7)&(2)

$x_1+x_2=10$

$x_1-x_2= 6$

$2x_1=16$

$x_1=8$

$x_2=2$

The intersection point is (8,2)

This point lies in the feasible region and satisfy both the constraint.

Hence the optimum solution is $x_1=8$, $x_2=2$ and max Z=32

Constrained Optimization

The problem of optimizing a *continuous and differential* function subject to constraints may be defined as:

Optimize $Z=f(x_1,x_2,...x_n)$

Subject to the constraints

$g_i(x_1,x_2,....x_n)\{<=, >=, =\}$ b_i, for $i=1,2,3....m$

and $x_j>=0$; $j=1,2,3...n$.

Note

Constrained optimization in matrix notation as follows

Optimize (Max or Min) $z=f(x)$

subject to constraints

$h_i(x)$ $\{<=,>=,=\}$ 0, and $x>=0$ for all $i=1,2,3...m$

where $h_i(x)=g_i(x)-b_i$

To solve such type of problem, we make use of Lagrange Multiplier method (in general Kuhn tucker conditions)

Note:-

There are two methods of solution used for solving NLP

(I) Kuhn Tucker Method or Lagrange's Multipliers method

(II) Beale's method

(III) Wolf Method

Kuhn Tucker method of solution to NLP

Kuhn-Tucker Conditions

The necessary and sufficient Kuhn-Tucker conditions to get an optimal solution to the problem of maximizing the given quadratic objective function subject to linear constraints can be derived as follows:

Step 1 Introducing slack variables s_i^2 and r_j^2 to constraints (1) and (2), the problem becomes

$$\text{Max } f(\mathbf{x}) = \sum_{j=1}^{n} c_j x_j - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_j d_{jk} x_k$$

subject to the constraints

$$\sum_{j=1}^{n} a_{ij} x_j + s_i^2 = b_i ; \quad i = 1, 2, \ldots, m$$

$$-x_j + r_j^2 = 0 \qquad ; \ j = 1, 2, \ldots, n$$

Step 2 Forming the Lagrange function as follows:

$$L(\mathbf{x}, \mathbf{s}, \mathbf{r}, \lambda, \mu) = f(\mathbf{x}) - \sum_{j=1}^{n} \lambda_i \{a_{ij} x_j + s_i^2 - b_i\} - \sum_{j=1}^{n} \mu_j \{-x_j + r_j^2\}$$

Step 3 Differentiate $L(\mathbf{x}, \mathbf{s}, \mathbf{r}, \lambda, \mu)$ partially with respect to the components of $\mathbf{x}$, $\mathbf{s}$, $\mathbf{r}$, λ and μ. Then equate these derivatives to zero to get the required Kuhn-Tucker necessary conditions. That is,

(i) $\mathbf{c} - \dfrac{1}{2}(2\,\mathbf{x}^T \mathbf{D}) - \lambda \mathbf{A} + \mu = 0$

or $c_j - \sum_{k=1}^{n} x_k d_{jk} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j = 0 ; \quad j = 1, 2, \ldots, n$

(ii) $-2\lambda\,\mathbf{s} = 0$ or $\lambda_i s_i^2 = 0$

or $\lambda_i \left\{ \sum_{j=1}^{n} a_{ij} x_j - b_i \right\} = 0, \quad i = 1, 2, \ldots, m$

(iii) $-2\mu\ r = 0$ or $\mu_j r_j = 0; \quad j = 1, 2, \ldots, n$

$\mu_j x_j = 0, \ j = 1, 2, \ldots, n$

(iv) $\mathbf{A}\mathbf{x} + \mathbf{s}^2 - \mathbf{b} = 0$; i.e. $\mathbf{A}\mathbf{x} \le \mathbf{b}$

or $\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m$

(v) $-\mathbf{x} + \mathbf{r}^2 = 0$, i.e. $\mathbf{x} \ge 0$

or $x_j \ge 0, \quad j = 1, 2, \ldots, n$

(vi) $\lambda_i, \ \mu_j, x_j, s_i, r_j \ge 0$

These conditions except (ii) and (iii) are linear constraints involving $2(n+m)$ variables. The condition $\mu_j x_j = \lambda_i s_i = 0$ implies that both x_j and μ_j as well as s_i and λ_i cannot be basic variables at a time in a non-degenerate basic feasible solution. The conditions $\mu_j x_j = 0$ and $\lambda_i s_i = 0$ are also called *complementary slackness conditions.*

Problem:-01

Use Graphical Method to Minimum the Distance of the Origin from the Convex region bounded by the following Constraint.

$x_1 + x_2 \geq 4$

$2x_1 + x_2 \geq 5$

$x_1, x_2 \geq 0$

Verify Kuhn-Tuber Necessary Condition Hold at the point of Minimum Distance.

Solution:-

Let us Draw The Line,

$x_1 + x_2 = 4$ --------1

$x_1 = 0$, $x_2 = 4$

$x_2 = 0$, $x_1 = 4$

The equation (1) is the line joining the points (0,4)& (4,0)

Let us Draw the Line

$2x_1 + x_2 = 5$ -------2

$x_1 = 0$ $x_2 = 5$

$x_2 = 0$ $x_1 = 2.5$

The equation (2) is the line joining the points (0,5)& (2.5,0)

To find the intersection points

Solve 1 & 2:

$x_1 + x_2 = 4$

$2x_1 + x_2 = 5$

$-x_1 = -1$

$x_1 = 1$

$x_2 = 3$

The intersection point is (1,3)

The feasible region is a convex bounded region, the problem is of minimizing the distance of solution point from the origin . i.e we have to minimum the radius of the circle whose center 0 and touching the feasible region.

Therefore the objective function is

Min $z = x_1^2 + x_2^2$

Gradient of objective function with respect to x_1

$x_1^2 + x_2^2 = 0$

$2x_1 + \frac{dx_2^{\,2}}{dx_1} = 0$

$2x_1 + \frac{dx_2^{\,2}}{dx_2} \cdot \frac{dx_2}{dx_1} = 0$

$2x_1 + 2x_2 \frac{dx_2}{dx_1} = 0$

$\frac{dx_2}{dx_1} \cdot \frac{-x_1}{x_2}$ --------3

Gradient Line

$x_1 + x_2 = 4$

$1 + \frac{dx_2}{dx_1} = 0$

$\frac{dx_2}{dx_1} = -1$ -------- 4

Gradient Line

$2x_1 + x_2 = 5$

$2 + \frac{dx_2}{dx_1} = 0$

$\frac{dx_2}{dx_1} = -2 - - - - - 5$

Sub 4 in 3:

$-1 = \frac{x_1}{x_2}$

$-x_2 + x_2 = 0$ -----------6

Solve 5 & 7:

$x_1 - x_2 = 0$

$x_1 + x_2 = 4$

$2x_2 = 4$

$x_2 = 2$

$x_1 = 2$

The solution point is (2,2)

Sub 5 in 3

$-2 = \frac{-x_1}{x_2}$

$-2x_2 + x_1 = 0$ ---------- 7

Solve 7 & 2 :

$-2x_2 + x_1 = 0$

$x_1 + 2x = 10$

$5x_1 = 10$

$x_1 = 2$

$x_2 = 1$

The solution point is (2,1)

Clearly the solution point (2,1)does not satisfies the first constraint in the given problem, $\therefore$ (2,1) does not lines in feasible region.

Clearly the solution point (2,2) satisfies all the constraints in the given problem, therefore (2,2) lies in the feasible region.

Verification of Kuhn Tucker Conditions

Since Min f (x) = $x_1^2 + x_2^2$

$x_1 + x_2 = 4$,

$\Rightarrow x_1 + x_2 - 4 = 0$

Let $h_1(x) = x_1 + x_2 - 4$

$2x_1 + x_2 = 5$

$2x_1 + x_2 - 5 = 0$

Let $h_2(x) = 2x_1 + x_2 - 5$

Kuhn-tucker Condition for Minimization NLP

Condition -I

$$\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{2} \lambda_i \frac{\partial h_i(x)}{\partial x_j} = 0 \quad for \quad \therefore 1,2$$

For j = 1,

$$\frac{\partial f(x)}{\partial x_1} - \sum_{i=1}^{2} \lambda_i \frac{\partial h_i(x)}{\partial x_1} = 0$$

$$\frac{\partial (x_1{}^2 + x_2{}^2)}{\partial x_1} - \left\{ \lambda_1 \frac{\partial h_1(x)}{\partial x_1} + \lambda_2 \frac{\partial h_2(x)}{\partial x_1} \right\} = 0$$

$$2x_1 - \left\{ \lambda_1 \frac{\partial (x_1 + x_2 - 4)}{\partial x_1} + \lambda_2 \frac{\partial (2x_1 + x_2 - 5)}{\partial x_1} \right\} = 0$$

$$2x_1 - \{\lambda_1 + 2\lambda_2\} = 0$$

$2x_1 - \lambda_1 - 2\lambda_2 = 0$ --------------(1)

For j =2,

$$\frac{\partial f(x)}{\partial x_2} - \sum_{i=1}^{2} \lambda_i \frac{\partial h_i(x)}{\partial x_2} = 0$$

$$\frac{\partial (x_1^2 + x_2^2)}{\partial x_2} - \left\{ \lambda_1 \frac{\partial h_1(x)}{\partial x_2} + \lambda_2 \frac{\partial h_1(x)}{\partial x_2} \right\} = 0$$

$$2x_2 - \left\{ \lambda_1 \frac{\partial (x_1 + x_2 - 4)}{\partial x_2} + \lambda_2 \frac{\partial (2x_1 + x_2 - 5)}{\partial x_2} \right\} = 0$$

$$2x_2 - \{\lambda_1 + \lambda_2\} = 0$$

$2x_1 - \lambda_1 - \lambda_2 = 0$ ------------------- (2)

Condition -II.

$\lambda_i \, h_i(x) = 0$, for i = 1,2

For i = 1

$\lambda_1 h_1(x) = 0$

$\lambda_1 (x_1 + x_2 - 4) = 0$ -------------------- (3)

For i = 2

$\lambda_2 h_2(x) = 0$

$\lambda_2 (2x_1 + x_2 - 5) = 0$ ------------------- (4)

Condition -III.

$h_i(x) \geq 0$ fori=1,2.

For i = 1

$h_1(x) \geq 0$

$x_1 + x_2 - 4 \geq 0$ -------------------------- (5)

For $i = 2$

$h_2(x) \geq 0$

$2x_1 + x_2 - 5 \geq 0$ ------------------------ (6)

Conditions -IV.

$\lambda_I \geq 0$, for $I = 1,2$

$\lambda_I \geq 0$, -------------------(7)

$\lambda_2 \geq 0$ --------------------(8)

Solve (1),(2)

$2x_1 - \lambda_1 - 2\lambda_2 = 0$ [sub $x_1 = 2$, $x_2 = 2$ Optimum]

$2x_2 - \lambda_1 - \lambda_2 = 0$

$\lambda_1 + 2\lambda_2 = 4$

$\lambda_1 + \lambda_2 = 4$

$2\lambda_2 = 0$

$\lambda_2 = 0$

$\lambda_1 = 4$

Therefore $(\lambda_1, \lambda_2) = (4,0)$

Therefore the solution $(x_1, x_2, \lambda_1, \lambda_2) = (2,2,4,0)$ satisfies all the equation from (1) –(8).

Hence the Kuhn-tucker conditions are satisfied at the optimal solution point (2,2).

The solution so obtained ... [cut-off line at top of page]

Beale's Method

In this method, instead of Kuhn-Tucker conditions, results based on calculus are used for solving a given quadratic programming problem. Let the general quadratic programming (QP) problem be of the form

$$\text{Minimize } Z = \mathbf{cx} + \frac{1}{2}\mathbf{x}^T \mathbf{D}\mathbf{x} \tag{3}$$

subject to the constraints: $\mathbf{Ax} = \mathbf{b}$; and $\mathbf{x} \geq 0$ (4)

where, $\mathbf{x} \in \mathbf{E}^n$, $\mathbf{b} \in \mathbf{E}^m$, $\mathbf{c} \in \mathbf{E}^n$, $\mathbf{D}$ is a symmetric $n \times n$ matrix and A is an $m \times n$ matrix.

Beale's method starts with partitioning of n variables in QP problem into *basic* and *non-basic variables* at each iteration of the solution process, and expressing the basic variables as well as objective function in terms of non-basic variables. Let $\mathbf{B}$ be any $m \times m$ non-singular matrix which contains columns of $\mathbf{A}$ corresponding to the basic variables $\mathbf{x}_B \in \mathbf{E}^m$ and let N be an $m \times (n - m)$ matrix which contains columns of $\mathbf{A}$ corresponding to non-basic variables $\mathbf{x}_N \in \mathbf{E}^{n-m}$. Then Eq. (5) can be written as

$$[\mathbf{B}, \mathbf{N}] \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} = \mathbf{b}$$

or

$$\mathbf{Bx}_B + \mathbf{Nx_N} = \mathbf{b} \quad \text{or} \quad \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Nx_N}$$

or

$$x_{B_i} = y_{i0} - \sum_{j=1}^{n-m} y_{ij} x_{N_j}; \; i = 1, 2, ..., m \tag{5}$$

where $\quad y_{i0} = (y_{10}, y_{20}, \ldots, y_{m0})^T = \mathbf{B}^{-1}\mathbf{b}$ and $y_{ij} = \mathbf{B}^{-1}\mathbf{N}$.

For the current basic feasible solution $x_{N_j} = 0$ $(j = 1, 2, \ldots, n - m)$, we have $x_{B_i} = y_{i0}$, $(i = 1, 2, \ldots, m)$. Assuming that $y_{i0} \geq 0$.

The objective function (3) in terms of $\mathbf{x}_B$ and $\mathbf{x}_N$ can be written as

$$Z = [\,\mathbf{c}_B, \mathbf{c}_N] \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} + \frac{1}{2}\left[\mathbf{x}_B^T, \mathbf{x}_N^T\right] \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix}$$

Expressing Z in terms of the remaining $(n - m)$ non-basic variables $\mathbf{x}_N$ only and simplifying we get

$$Z = Z_0 + \alpha \mathbf{x}_N + \mathbf{x}_N^T\, \mathbf{G}\mathbf{x}_N \tag{6}$$

where $\quad Z_0$ = value of objective function Z when $\mathbf{x}_N = 0$ and $x_{Bi} = y_{i0}$

$\mathbf{G}$ = symmetric matrix of order $(n - m) \times (n - m)$

$\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{n-m}$ (constant)

Step 1 Evaluate the partial derivatives of Z with respect to non-basic variables $\mathbf{x}_N = 0$. Thus from Eq. (6) we get

$$\frac{\partial Z}{\partial x_{Nj}} = \alpha_j + 2\sum_{k=1}^{n-m} g_{jk}\, x_{N_k}; \quad j = 1, 2, \ldots, n - m$$

Step 2 See the nature of $\left|\dfrac{\partial Z}{\partial x_{Nj}}\right|_{x_N = 0} = \alpha_j; \quad k = 1, 2, \ldots, n - m$

(a) If $\alpha_j < 0$, for all j, then the current solution is also an optimal solution.

(b) But if at least one $\alpha_j > 0$, then one of the non-basic variables which is currently at zero level corresponding to largest positive value of α_j, will be selected to enter into the basis.

Step 3 $\left.\dfrac{\partial Z}{\partial x_N}\right|_{x_N=0} = \alpha_r$ (largest), then choose non-basic variable x_r for entering into the basis. For this it will be profitable to go on increasing its value from zero to a point, till either

(a) any one of the present basic variables becomes negative, or

(b) $\partial Z/\partial x_{Nj}$ reduces to zero and is about to become negative.

Step 4 For maintaining the feasibility of the solution we must consider only that value of non-basic variable x_r, say β_1, which has only a positive coefficient. In this case, the first basic variable selected to leave the basis should satisfy the usual minimum ratio rule of the simplex method and is given by

$$\beta_1 = \begin{cases} \text{Min}\left\{\dfrac{y_{i0}}{y_{ij}};\ y_{ij} > 0\right\} \\ \infty \qquad\qquad ;\ y_{ij} \le 0,\ \ j = 1, 2, \dots,\ n-m \end{cases}$$

where $y_{i0} = x_{Bi}$.

Since it is not desirable to increase the value of the non-basic variable x_r beyond the point where $\partial Z/\partial x_{N_j}$ becomes zero, the critical value of x_r say β_2, at which $\partial Z/\partial x_{N_j}$ becomes zero is given by

$$\beta_2 = \begin{cases} \dfrac{|\alpha_j|}{2g_{jj}} ; & g_{jj} > 0 \\ \infty ; & g_{jj} \le 0 \end{cases}$$

where g_{jj} is the element of matrix **G**.

Hence, the value of non-basic variable x_r must be determined by taking the minimum between β_1 and β_2, that is, $x_r = \text{Min}\ \{\beta_1, \beta_2\}$

However, if $\beta_1 = \beta_2 = \infty$, the value of x_r can be increased indefinitely without violating either of the conditions (a) and (b) of Step 3 and the QP problem must have an unbounded solution. Moreover

(i) If the entering variable x_r is increased up to β_1 only and at least one basic variable is reduced to zero, then a new basic feasible solution can be obtained by the usual simplex method. But if by entering x_r into the basis two or more basic variables reduced to zero, then the new solution so obtained will be degenerate and thus cycling can occur.

(ii) If the entering variable is increased up to $\beta_2\ (< \infty)$, then we may have more than m variables at positive level at any iteration. This stage comes when the new (non-basic) feasible solution occur where $\partial Z/\partial x_{Nj} = 0$. At this stage we define a new variable (unrestricted) u_j as

$$u_j = \frac{\partial Z}{\partial x_r} = \alpha_j + 2\sum_{k=1}^{n-m} g_{jk}\, x_{Nk}$$

The variable u_j is also called *free variable*. Clearly, we now have $m + 1$ non-zero variables and $m + 1$ constraints. These variables form a basic feasible solution to the new set of constraints:

$$\mathbf{Ax} = \mathbf{b}$$

$$u_j - 2\sum_{k=1}^{n-m} g_{jk}\, x_{Nk} = \alpha_j$$

The variable u_j is introduced in the set of constraints only for computational purposes and its value is zero at the next basic feasible solution. Now the variables $\mathbf{x}_B$ and u_j are treated as basic variables. The new set of constraints is again expressed in terms of non-basic variables to get the new basic feasible solution.

Step 5 Go to Step 1 and the entire procedure of getting a new basic feasible solution is repeated until no further improvement in the objective function may be obtained by making any permitted changes in one of the non-basic variables. The permitted changes here include increase in all variables and decrease in free variables. In other words, the procedure terminates when

$$\frac{\partial Z}{\partial x_{Nj}} \begin{cases} \leq 0, & \text{if } x_{Nj} \text{ is a restricted (non-negative) variable} \\ = 0, & \text{if } x_{Nj} \text{ is a free variable.} \end{cases} \quad (7)$$

The necessary conditions (8) for terminating the procedure are also sufficient for a global minimum if D is positive semi-definite or positive definite.

Remarks 1. While evaluating $\partial Z/\partial u_j$, both increase and decrease must be checked, as u_j is unrestricted in sign.

2. If at any iteration a free variable becomes a basic variable and is non-zero, then drop the new constraint containing it. This should be done because it is a free variable and therefore will neither be chosen to leave the basis nor will appear in the selection of leaving variable.

Problem:-01

Use Beale's method to solve quadratic programming problem.

Max $Z = 2x_1+3x_2-2x_2^2$

subject to

$x_1+4x_2 \leq 4$

$x_1+x_2 \leq 2$

and $x_1, x_2 \geq 0$

Solution:-

Let us add non-negative slack variables s_1, s_2 in the given constraints

$x_1+4x_2+S_1 = 4$

$x_1+x_2+S_2=2$

Initial Basic Feasible Solution is

$S_1=4$

$S_2=2$

$X_B=(s_1,s_2)=(4,2)$ [Basic variables]

$X_N=(x_1,x_2)=(0,0)$ [Non basic variables]

Y_B	X_B	x_1	x_2	S_1	S_2
S_1	4	1	4	1	0
S_2	2	1	1	0	1

Let us rewrite constraints terms of non basic variable

From the first row in the table

$S_1=4-x_1-4x_2$ [x_1, x_2 are NB variables, and S_1, S_2 are Basic variables]

From the second in the table

$S_2=2-x_1-x_2$

Let us rewrite the objective function in terms of non basic variables

$Z=2x_1+3x_2-2x_2^2$

Let us calculate partial derivative of Z with respect to non-basic variable x_1

$$\frac{\partial z}{\partial x_1}=2$$

Let us calculate partial derivative of Z with respect to non-basic variable x_2

$$\frac{\partial z}{\partial x_2}=3-4x_2$$

$$\left(\frac{\partial z}{\partial x_1}\right)_{x_1=0,x_2=0}=2=\alpha_1$$

i.e there will be increase in z when x_1 increases

$$\left(\frac{\partial z}{\partial x_2}\right)_{x_1=0,x_2=0} = 3 = \alpha_2$$

i.e there will be increase in z when x_2 increases

Thus there is greater increase in z with respect to increase in x_2 as compared to increase in x_1.

Rule:-

If (α_1) & (α_2)<0, optimal solution,

if at least one greater than 0, non-optimal solution .

Since both are positive , therefore the solution is not optimal.

Let us choose the variable to enter the basis

We choose x_2 to enter into the basic (due to most positive value of alpha) to improve the value of objective function.

Y_B	X_B	x_1	x_2	S_1	S_2	$\beta_1 = X_B / y_r$
S_1	4	1	**4**	1	0	4/|4|=1
S_2	2	1	**1**	0	1	2/|1|=2

To choose the variable leave the basis

I. $$\beta_1 = \begin{cases} \min\left\{\frac{x_{Bi}}{y_{ij}}\right\} & y_{ij} > 0 \\ \min\left\{\frac{x_{Bi}}{|y_{ij}|}\right\} & y_{ij} < 0 \end{cases}$$

=min{1,2}

=1

Since it is not desirable to increase the value of the non basic variable x_2 beyond the point where $\frac{\partial z}{\partial x_2}$ become zero.

$3-4x_2=0$

i.e x_2 can not be increased beyond 3/4, otherwise x_2 will be negative.

$$\beta_2 = \frac{3}{4}$$

A critical value of x_2 is given by

$x_2 = \min\{\beta_1, \beta_2\}$

$=\min\{1,3/4\}$

$x_2=3/4$ (corresponding to β_2)

$x_2=\beta_2$

Therefore case II has to be applied

We have to introduce the new free variable its value is zero in the next feasible solution.

Let us introduce free variable u_1 (unrestricted) and the corresponding constraint is $u_1=3-4x_2$ [x_2 enters the basis, therefore $u_1=\frac{\partial z}{\partial x_2}$]

$4x_2+u_1=3$,

We have to add this new constraint in the given problem

Y_B	X_B	X_1	x_2	S_1	S_2	u_1
S_1	4	1	**4**	1	0	0
S_2	2	1	**1**	0	1	0
u_1	**3**	**0**	**[4]**	**0**	**0**	**1**

Since x_2 enter the basis and u_1 leaves the basis. Use usual simplex calculation gives the following table values.

Y_B	X_B	X_1	X_2	S_1	S_2	u_1
S_1	**1**	**1**	**0**	**1**	**0**	**-1**
S_2	5/4	**1**	0	0	1	-1/4
x_2	¾	**0**	1	0	0	+1/4

In the column u_1 multiply the values by -1 [this is only when we remove free variable from the last row, not for all table]

Y_B	X_B	X_1	X_2	S_1	S_2	u_1
S_1	**1**	**1**	**0**	**1**	**0**	**+1**
S_2	5/4	**1**	0	0	1	+1/4
x_2	¾	**0**	1	0	0	-1/4

Check for optimality

$X_B=(s_1,s_2,x_2)=(1,5/4,3/4)$ [Basic variables]

$X_N=(x_1,u_1)=(0,0)$ [Non basic variables]

Let us rewrite the constraints in non basic variables

From the above table first row

$x_1+s_1+u_1=1$

write the B.V in-terms of N.B.V

$s_1=1-x_1-u_1$

Similarly from the above table second row

$x_1+s_2+\frac{1}{4}u_1=\frac{5}{4}$

write the B.V in-terms of N.B.V

$s_2=\frac{5}{4}-x_1-\frac{1}{4}u_1$

Also from the above table third row

$$x_2 - \frac{1}{4}u_1 = \frac{3}{4}$$

write the B.V in-terms of N.B.V

$$x_2 = \frac{3}{4} + \frac{1}{4}u_1$$

Let us rewrite the objective function in-terms of non basic variable

$Z = 2x_1 + 3x_2 - 2x_2^2$

$$z = 2x_1 + 3\left(\frac{3}{4} + \frac{u_1}{4}\right) - 2\left(\frac{3}{4} + \frac{u_1}{4}\right)^2$$

$$Z = \frac{9}{8} + 2x_1 - \frac{u_1^2}{8}$$

Let us partially differentiate Z with respect to non basic variables x_1 and u_1

$$\frac{\partial z}{\partial x_1} = 2 \qquad \frac{\partial z}{\partial u_1} = \frac{-2u_1}{8} = \frac{-u_1}{4}$$

$$\left(\frac{\partial z}{\partial x_1}\right)_{x_1=0,u_1=0} = 2 = \alpha_1 \qquad \left(\frac{\partial z}{\partial u_1}\right)_{x_1=0,u_1=0} = \frac{-0}{4} = 0 = \alpha_2$$

$\alpha_1 > 0$ .i.e there will be an increment in z when x_1 increases

$\alpha_2 = 0$ i.e there is no change in z when u_1 increase.

Since $\alpha_1 > 0$, thus the above solution is not optimal.

Let us choose the variable enters the basis

$\alpha_1 = 2 > 0$ (corresponding to the variable x_1), therefore we choose x_1 to enters the basis

Y_B	X_B	X_1	X_2	S_1	S_2	u_1	$\beta_1 = X_B / y_r$
S_1	**1**	**1**	0	1	0	+1	1/|1|=1
S_2	**5/4**	**1**	0	0	1	+1/4	(5/4)/|1|=5/4
x_2	**¾**	**0**	1	0	0	-1/4	-

To find the variable leave the basis

I. $\beta_1 = \min\left\{\frac{1}{1}, \frac{5}{4}\right\}$

=1 (corresponding to s_1)

II. If $\frac{\partial z}{\partial x_1} = 0$, then the maximum value of x_1 can be obtained, and $\beta_2 = x_1$

Here $\frac{\partial z}{\partial x_1} = 2 \neq 0$

Let $\beta_2 = \frac{\partial z}{\partial x_1} = 2$

$x_1 = \min\{\beta_1, \beta_2\}$

=min{1,2}

=1

$x_1 = \beta_1$ (corresponding variable s_1)

Thus case I has to be applied

Therefore s_1 leaves the basis

Y_B	X_B	X_1	X_2	S_1	S_2	u_1
S_1	**1**	**[1]**	**0**	**1**	**0**	**+1**
S_2	5/4	**1**	0	0	1	+1/4
x_2	¾	**0**	1	0	0	-1/4

Usual simplex procedure apply

Y_B	X_B	X_1	X_2	S_1	S_2	u_1
x_1	1	1	0	1	0	1
S_2	¼	0	0	-1	1	-3/4
x_2	¾	0	1	0	0	-1/4

[Note Here do not multiply the last column $u_1$by -1]

Check for optimality again

$X_B=(x_1,s_2,x_2)=(1,1/4,3/4)$ [Basic variables]

$X_N=(s_1,u_1)=(0,0)$ [Non basic variables]

From the first row in the above table

$x_1+s_1+u_1=1$

$x_1=1- s_1-u_1$

From the second row in the above table

$-s_1+s_2-\frac{3}{4}=\frac{1}{4}$

$s_2=\frac{1}{4}+\frac{3}{4}+ s_1$

From the third row in the above table

$$x_2-\frac{1}{4}u_1=\frac{3}{4}$$

$x_2=\frac{3}{4}+\frac{1}{4}u_1$

objective function

$Z=2x_1+3x_2-2x_2^2$

$$z=\frac{25}{8}-2s_1-2u_1-\frac{1}{8}u_1^2$$

Let us partially differentiate with respect to s_1 and u_1

$$\frac{\partial z}{\partial s_1}=-2, \qquad \frac{\partial z}{\partial u_1}=-2$$

Since $\alpha_1<0$, $\alpha_2<0$

Therefore the solution is optimal

Hence the optimal solution is

$x_1=1$, $x_2=3/4$ and max $z=25/8$.

Note:-

Since $X_B=(x_1, x_2)=(1,3/4)$ and $X_{NB}=(s_1, u_1)=(0,0)$

$$z=\frac{25}{8}-2s_1-2u_1\frac{1}{8}u_1^2=\frac{25}{8}-2(0)-2(0)-\frac{1}{8}(0)^2=25/8$$

Wolfe's Modified Simplex Method

Step 1 Introduce artificial variables A_j ($j = 1, 2, \ldots, n$) in the Kuhn-Tucker condition (i). Then we have

$$c_j - \sum_{k=1}^{n} x_k d_{jk} - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_j + A_j = 0$$

For a starting basic feasible solution, we shall have $x_j = 0$, $\mu_j = 0$, $A_j = -c_j$ and $s_i^2 = b_i$. However, for any real problem, this solution would be desirable if and only if $A_j = 0$ for all j.

Step 2 Apply Phase I of the simplex method to check the feasibility of the constraints $\mathbf{Ax} \leq \mathbf{b}$. If there is no feasible solution, then terminate the solution procedure, otherwise get an initial basic feasible solution for Phase II. To obtain the desired feasible solution solve the following problem:

$$\text{Minimize } Z = \sum_{j=1}^{n} A_j$$

subject to the constraints

$$\sum_{k=1}^{n} x_k d_{jk} + \sum_{i=1}^{m} \lambda_i a_{ij} - \mu_j + A_j = -c_j; \quad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} a_{ij} x_j + s_i^2 = b_i; \quad i = 1, 2, \ldots, m$$

and $\lambda_i, x_j, \mu_j, s_i, A_j \geq 0$ for all i and j

$$\left.\begin{aligned} \lambda_i s_i &= 0 \\ \mu_i x_i &= 0 \end{aligned}\right\} \text{Complementary slackness conditions}$$

Thus, while deciding for a variable to enter into the basis at each iteration, the complementary slackness conditions must be satisfied. This problem has $2(m + n)$ variables and $(m + n)$ linear constraints together with $(m + n)$ complementary slackness conditions.

Step 3: Apply Phase II of the simplex method to get an optimal solution to the problem given in Step 2. The solution so obtained will also be an optimal solution of the quadratic programming problem.

Note:-(Simple procedure as follow)

Step:-01

Write the given quadratic programming problem in standard form by adding quadratic slack variables .

Step:-02

Form the Lagrange's function for finding Kuhn tucker conditions and get set of equations.

Step:-03

Add artificial variable in the new set of equation and these becomes new constraints for the newly formed LPP

Create the new objective function using artificial variables.

The collection of new objective function and constraints form new LPP.

Step:-04

Apply two phase simplex method and check for optimality.

Problem:-01

Use wolf method to solve quadratic programming problem

Max $Z=4x_1+6x_2-2x_1^2-2x_1x_2-2x_2^2$

Subject to constrain

$x_1+2x_2\leq 2$

$x_1,x_2\geq 0$

Solution:-

Step:-01

Note:-

(i) Convert the objective function in to maximization form

Max $Z=4x_1+6x_2-2x_1^2-2x_1x_2-2x_2^2$, it is already in maximization form

(ii) Write all the constraints in '<=' type

$x_1+2x_2\leq 2$, it is already in <= type

Let us convert the non - negative constraints as follows

$-x_1<=0$

$-x_2<=0$

The standard form of quadratic LPP

$x_1+2x_2+q_1^2=2$ [q_1^2 is quadratic slack variable]

Newly added constrains are

$-x_1+r_1^2=0$ and $-x_2+r_2^2=0$ [r_1^2, r_2^2 are quadratic slack variable]

and x_1, x_2, q_1^2 , r_1^2, $r_1^2 \geq 0$

Step:-02

We construct the Lagrange's function $L(x,q,r,\lambda,\mu)$

Here λ,μ are called Lagrange's multiplier or constant.

$$L(x_1,x_2,q_1,r_1,r_2,\lambda_1,\mu_1,\mu_2)= 4x_1+6x_2-2x_1^2-2x_1x_2-2x_2^2-\lambda_1(x_1+2x_2+q_1^2-2)$$
$$-\mu_1(-x_1+r_1^2)-\mu_2(-x_2+r_2^2)$$

The necessary and sufficient condition for Kuhn tucker are

$$\frac{\partial}{\partial x_1}(L)=0$$

$$\Rightarrow 4-4x_1-2x_2-\lambda_1+\mu_1=0$$

$$\frac{\partial}{\partial x_2}(L)=0$$

$$\Rightarrow 6-2x_1-4x_2-2\lambda_2+\mu_2=0$$

$$\frac{\partial L}{\partial q_1}=0$$

$$\Rightarrow \lambda_1 q_1=0$$

Multiply both sides by q_1

$$\Rightarrow \lambda_1 q_1^2=0$$

$$\Rightarrow \lambda_1 s_1=0$$ [Assume $s_1=q_1^2$]

$$\frac{\partial L}{\partial \lambda_1}=0$$

$$\Rightarrow x_1+2x_2+q_1^2-2=0$$

$$\frac{\partial L}{\partial \mu_1}=0$$

$$\Rightarrow -x_1+r_1^2=0$$

$$\Rightarrow x_1=r_1^2$$

$$\frac{\partial L}{\partial \mu_2}=0$$

$$=> -x_2 + r_1^2 = 0$$

$$=> x_2 = r_1^2$$

$$\frac{\partial L}{\partial r_1} = 0$$

$$=> 2\mu_1 r_1 = 0$$

Multiply both by r_1 and divide by 2

$$=> \mu_1 r_1^2 = 0$$

$$=> \mu_1 x_1 = 0 \qquad [\because x_1 = r_1^2]$$

$$\frac{\partial L}{\partial r_2} = 0$$

$$=> 2\mu_2 r_2 = 0$$

Multiply both sides by r_2 and divide by 2

$$=> \mu_2 r_2^2 = 0$$

$$=> \mu_2 x_2 = 0 \qquad [\because x_2 = r_2^2]$$

Thus new set of equations(Kuhn tucker conditions) are

$4x_1 + 2x_2 + \lambda_1 - \mu_1 = 4$ -------------(1)

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 = 6$ ------------(2)

$x_1 + 2x_2 + s_1 = 2$ -------(3) [Here $s_1=q_1^2$]

$\{\lambda_1 s_1 = 0, \mu_1 r_1 = 0, \mu_2 r_2 = 0\}$ ----------(4) (this is called complimentary conditions)

and x_1, x_2, $\lambda_1, \mu_1, \mu_2, s_1 \geq 0$

Step:-03

Introduce artificial variable A_1 and A_2 in the first two constraints respectively and form new objective function .

The modified LP problem becomes

Max $Z^*=-A_1-A_2$

Add artificial variable only if it does not contain quadratic slack/surplus variable.

$4x_1 + 2x_2 + \lambda_1 - \mu_1 + A_1 = 4$ [it does not contain slack/surplus variable]

$2x_1 + 4x_2 + 2\lambda_1 - \mu_2 + A_2 = 6$ [it does not contain slack/surplus variable]

$x_1 + 2x_2 + s_1 = 2$ [No need to add, it contain slack variable s_1]

where $x_1, x_2, \lambda_1, \mu_1, \mu_2, A_1, A_2, s_1 \geq 0$

Initial basic feasible solution of LPP

Let $x_1=x_2=\lambda_1=\mu_1=\mu_2=0$, we get $A_1=4, A_2=6$, $s_1=2$,

Therefore Max $Z^*=-4-6=-10$

Initial simplex table

	C_j	0	0	0	0	0	0	-1	-1
C_B	X_B	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_1	A_2
-1	**$A_1=4$**	**[4]**	**2**	**1**	**-1**	**0**	**0**	**1**	**0**
-1	$A_2=6$	**2**	4	2	0	-1	0	0	1
0	$s_1=2$	**1**	2	0	0	0	1	0	0
	Z_j	**6**	6	3	-1	-1	0	1	1
	Z_j-C_j	**-6**	-6	-3	1	1	0	0	0

Since some $Z_j - C_j <=0$, therefore the solution is not optimal

Rule :- (To select the variable enter into basis)

Here x_1, x_2 and λ_1 are eligible to enter the basis, λ_1 cannot enter the basis since the complementary condition $\lambda_1 s_1=0$ does not hold.

Since x_1 and x_2 equally competitive and complementary conditions

$x_1\mu_1=0$, $x_2\mu_2=0$ are satisfied

We select x_1 enters the basis

Rule :-(To select the variable leaves the basis)

We have to make all A_1, A_2 equal to zero, i.e it should become non basic variable. So first we select the variable A_1 leaves the basis.

Apply usual simplex procedure , we get

	C_j	0	0	0	0	0	0	-1
C_B	X_B	x_1	x_2	λ_1	μ_1	μ_2	S_1	A_2
0	$x_1=4$	1	½	1/4	-1/4	0	0	0
-1	$A_2=4$	0	**3**	3/2	1/2	-1	0	1
0	**$s_1=1$**	**0**	**3/2**	**-1/4**	**1/4**	**0**	**1**	**0**
	Z_j	0	3	3/2	1/2	-1	0	1
$Z^*=-4$	$Z_j - C_j$	0	-3	-3/2	-1/2	1	0	-1

[Note: Once A_1 leaves the basis, it could not enter basis again, the corresponding column need not be written.]

Since some $Z_j - C_j <= 0$, therefore the above solution is not optimal.

Rule :- (To select the variable enter into basis)

Here x_2, λ_1 and μ_1 are eligible to enter the basis,

λ_1 cannot enter the basis since the complementary condition $\lambda_1 s_1=0$ does not hold.

μ_1 cannot enter the basis since the complementary condition $x_1\mu_1=0$ does not hold.

x_2 can enter the basis and the complementary conditions $x_2\mu_2=0$ hold

We select x_2 to enters the basis.

Rule :-(To select the variable leaves the basis)

If we select A_2 as leaving variable, then in the next table s_1 remains in the basic variable column and C_j-Z_j value of λ_1 will be negative, so in the next table entering variable will be λ_1, but you can not enter since complementary condition $\lambda_1 s_1=0$ does not hold. i.e we stuck at this stage could not move further.

So we select s_1 as leaving variable.

	C_j	0	0	0	0	0	0	-1
C_B	X_B	x_1	x_2	λ_1	μ_1	μ_2	s_1	A_2
0	$x_1=2/3$	1	0	**1/3**	-1/3	0	-1/3	0
-1	**$A_2=2$**	**0**	**0**	**[2]**	**0**	**-1**	**-2**	**1**
0	$x_2=2/3$	0	1	**-1/6**	1/6	0	2/3	0
	Z_j	0	0	2	0	-1	-2	1
$Z^*=2$	Z_j-C_j	0	0	-2	0	1	2	0

Since $Z_3-C_3=-2<0$, therefore the above table is not optimal

Rule :- (To select the variable enter into basis)

Here λ_1 are eligible to enter the basis,

λ_1 can enter the basis since the complementary condition $\lambda_1 s_1=0$ does not hold.

We select λ_1 to enters the basis.

Rule :-(To select the variable leaves the basis)

We have only one variable to leave the basis A_2

So we select A_2 as leaving variable.

	C_j	0	0	0	0	0	0
C_B	X_B	x_1	x_2	λ_1	μ_1	μ_2	S_1
0	$x_1=1/3$	1	0	0	-1/3	-1/6	0
1	$\lambda_1=1$	0	0	1	0	-1/2	-1
0	$x_2=5/6$	0	1	**0**	1/6	-1/2	1/2
	Z_j	0	0	0	0	0	0
$Z^*=0$	Z_j-C_j	0	0	0	0	0	0

[Note: Once A_2 leaves the basis, it could not enter basis again, the corresponding column need not be written.]

Since all $C_j- Z_j>=0$, therefore the above solution is optimal

The optimum solution is

$x_1=1/3$, $x_2=5/6$, $\lambda_1=1$,$\mu_1=\mu_2=s_1=0$ Min $Z^*=0$

This solution also satisfies the complementary conditions λ_1 $S_1=0$, μ_1 $x_1=0$,

$\mu_2\ x_1=0$ and the restriction on the sign of Lagrange's multipliers λ_1, μ_1 and μ_2

The optimum solution for the given quadratic problem is

$x_1=1/3,\ x_2=5/6,\ \lambda_1=1, \mu_1=\mu_2=s_1=0$

Max $Z=4(1/3)+6(5/6)-2(1/3)^2-2(1/3)(5/60)-2(5/6)^2= 25/6$

Problem:-2

Apply Wolfe method to solve the quadratic programming problem

Max $Z=2x_1+x_2-x_1^2$

Subject to constrains

$2x_1+3x_2\leq 6$

$2x_1+x_2\leq 4$

and $x_1,x_2\geq 0$

Solution

The standard form of quadratic programming problem

Max $Z=2x_1+x_2-x_1^2$

$2x_1+3x_2+q_1^2=6$ [add quadratic slack variable q_1^2]

$2x_1+x_2+q_2^2=4$ [add quadratic slack variable q_2^2]

Newly added constraints are

$-x_1+r_1^2=0$ and $-x_2+r_2^2=0$ [add quadratic slack variables r_1^2 ,r_2^2]

and $x_1,\ x_2,\ q_1^2, q_2^2,\ r_1^2,\ r_1^2\geq 0$

We construct the Lagrange's function L (x,q,r,λ,μ)

$$L\ (x,q,r,\ \lambda\ ,\mu)=\ 2x_1+x_2-x_1^2-\ \lambda_1(2x_1+3x_2+q_1^2-6)-\ \lambda_2(2x_1+x_2+q_2^2-4)$$
$$-\ \mu_1(-x_1+r_1^2)-\ \mu_2(-x_2+r_2^2)$$

The necessary and sufficient conditions (Kuhn tucker conditions) are

$$\frac{\partial}{\partial x_1}(L)=0$$

$$=> 2-2x_1-2\lambda_1-2\lambda_2+\mu_1=0$$

$$\frac{\partial}{\partial x_2}(L)=0$$

$$=> 1-3\lambda_1-\lambda_2+\mu_2=0$$

$$\frac{\partial L}{\partial \lambda_1}=0$$

$$=> -2x_1+3x_2+q_1^{\ 2}-6=0$$ [Let $q_1^2=s_1$]

$$=> -2x_1+3x_2+s_1-6=0$$

$$\frac{\partial L}{\partial \lambda_2}=0$$

$$=> -2x_1+x_2+q_2^{\ 2}-4=0$$

$$=> -2x_1+x_2+s_2-4=0$$ [Let $q_2^2=s_2$]

$$\frac{\partial L}{\partial q_1}=0$$

$$=> -\lambda_1(2q_1)=0$$

Multiply both sides by q_1 and divide by 2

$$=> \lambda_1 q_1^{\ 2}=0$$

$$=> \lambda_1 s_1=0$$ [Since $q_1^2=s_1$]

$$\frac{\partial L}{\partial q_2}=0$$

$$=> -\lambda_2(2q_2)=0$$

Multiply both sides by q_2 and divide by 2

$$=> \lambda_2 q_2^{\ 2}=0$$

$$=> \lambda_2 s_2=0$$ [Since $q_2^2=s_2$]

$$\frac{\partial L}{\partial \mu_1}=0$$

$$=> -x_1+r_1^{\ 2}=0$$

$$\frac{\partial L}{\partial \mu_2}=0$$

$$=> -x_2+r_2^{\ 2}=0$$

$$\frac{\partial L}{\partial r_1}=0$$

$=> 2\mu_1 r_1 = 0$

Multiply both sides by r_1 and divide by 2

$=> \mu_1 r_1^{\ 2} = 0$

$=> \mu_1 x_1 = 0$ $\qquad [\because x_1 = r_1^{\ 2}]$

$\frac{\partial L}{\partial r_2} = 0$

$=> -2\mu_2 r_2 = 0$

Multiply both sides by r_2 and divide by 2

$=> \mu_2 r_2^{\ 2} = 0$

$=> \mu_2 x_2 = 0$ $\qquad [\because x_2 = r_2^{\ 2}]$

The kuhn tucker conditions are

$2x_1 + 2x_2 + \lambda_1 - \mu_1 = 2$ -------------(1)

$3\lambda_1 + \lambda_2 - \mu_1 = 1$------------(2)

$2x_1 + 3x_2 + s_1 = 6$-------(3)

$2x_1 + x_2 + s_2 = 4$--------(4)

$\lambda_1 q_1 = \lambda_2 q_2 = \mu_1 r_1 = \mu_2 r_2 = 0$------------(5) (these are called complimentary conditions)

Max $Z^* = -A_1 - A_2$ where A_1, A_2 , are artificial variables

Subject to constraints

$2x_1 + 2\lambda_1 + 2\lambda_2 - \mu_1 + A_1 = 2$

$3\lambda_1 + \lambda_2 - \mu_1 + A_2 = 1$

$2x_1 + 3x_2 + s_1 = 6$

$2x_1 + x_2 + s_2 = 4$

$x_1, x_2, \lambda_1, \lambda_2, \mu_1, \mu_2, A_1, A_2, s_1, s_2 \geq 0$

Initial basis feasible solution is

$x_1 = x_2 = \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$

$A_1 = 2, A_2 = 1, s_1 = 6, s_2 = 4$

	C_j	0	0	0	0	0	0	0	0	-1	-1
C_B	x_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_1	A_2
-1	**A_1=2**	**[2]**	**0**	**2**	**2**	**-1**	**0**	**0**	**0**	**1**	**0**
-1	A_2=1	**0**	0	3	1	0	-1	0	0	0	1
0	s_1=6	**2**	3	0	0	0	0	0	0	0	0
0	s_2=4	**2**	1	0	0	0	0	0	0	0	0
	Z_j	2	0	5	3	-1	-1	0	0	1	1
Z^*=-3	Z_j-C_j	-2	0	-5	-3	1	1	0	0	-1	-1

Since some Z_j-C_j<=0, therefore the solution is not optimal

Here x_1 , x_2, λ_1, λ_2 are eligible to enter the basis

λ_1 can not enter the basis since complementary condition $\lambda_1 s_1=0$ does not hold

λ_2 can not enter the basis since complementary condition $\lambda_2 s_2=0$ does not hold

x_1 can enter basis since complementary condition $x_1\ \mu_1=0$ hold.

We select x_1 to enter the basis

We have select A_1 as leaving variable.

	C_j	0	0	0	0	0	0	0	0	-1
C_B	x_B	x_1	**x_2**	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2
0	x_1=1	1	**0**	1	1	-1/2	0	0	0	0
-1	A_2=1	0	**0**	3	1	0	-1	0	0	1
0	**s_1=4**	**0**	**[3]**	**-2**	**-2**	**1**	**0**	**0**	**0**	**0**
0	S_2=2	0	**1**	-2	-2	1	0	0	0	0
	Z_j	0	0	3	1	0	-1	0	0	1
Z^*=-1	Z_j -C_j	0	0	-3	-1	0	1	0	0	0

Since some Z_j-C_j<=0, therefore the solution is not optimal

Here λ_1, λ_2 x_2, μ_1 are eligible to enter the basis

λ_1 can not enter the basis since complementary condition $\lambda_1 s_1=0$ does not hold

λ_2 can not enter the basis since complementary condition $\lambda_2 s_2=0$ does not hold

μ_1 can not enter basis since complementary condition x_1 μ_1=0 does not hold.

x_2 can enter the basis since complementary condition x_2 μ_2=0 hold.

We select x_2 to enter the basis

We have select s_1 as leaving variable.

	C_j	0	0	0	0	0	0	0	0	-1
C_B	x_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2	A_2
0	x_1=1	1	0	**1**	1	-1/2	0	0	0	0
-1	**A_2=1**	**0**	**0**	**[3]**	**1**	**0**	**-1**	**0**	**0**	**1**
0	x_2=4/3	0	1	**-2/3**	-2/3	1/3	0	0	0	0
0	s_2=2/3	0	0	**-4/3**	-4/3	2/3	0	0	0	0
	Z_j	0	0	3	1	0	-1	0	0	1
Z^*=-1	Z_j C_j	0	0	-3	-1	0	1	0	0	0

Since some Z_j-C_j<=0, therefore the solution is not optimal

Here λ_1, λ_2 ,μ_1 are eligible to enter the basis

λ_1 can enter the basis since complementary condition $\lambda_1 s_1$=0 not hold

λ_2 can not enter the basis since complementary condition $\lambda_2 s_2$=0 does not hold

μ_1 can not enter basis since complementary condition x_1 μ_1=0 does not hold.

We select λ_1 to enter the basis

We have select A_2 as leaving variable.

	C_j	0	0	0	0	0	0	0	0
C_B	x_B	x_1	x_2	λ_1	λ_2	μ_1	μ_2	s_1	s_2
0	x_1=2/3	1	0	0	2/3	-1/2	1/3	0	0
0	λ_1=1/3	0	0	1	1/3	0	-1/3	0	0
0	x_2=2/3	0	1	0	-4/9	1/3	-2/9	0	0
0	s_2=10/9	0	0	0	-8/9	2/3	-4/9	0	0
	Z_j	0	0	0	0	0	0	0	0
Z^*=0	Z_j -C_j	0	0	0	0	0	0	0	0

Since all C_j-Z_j=0 , therefore the above solution is optimal

The optimal solution is $x_1=2/3$, $x_2=2/3$, $\lambda_1=1/3$, $\lambda_2=0$, $s_1=0$, $s_2=10/9$, $\mu_1=\mu_2=0$ and Max $Z^*=0$

This solution also satisfies the complementary condition $\lambda_1 s_1=0$, $\lambda_2 s_2=0$, $\mu_1 x_1=0$, $x_2\mu_2=0$ and restriction on the sign of Lagrange's multiples λ_1, λ_2, μ_1, μ_2.

The optimum solution of the given quadratic problem is given by

Max $Z=2x_1+x_2-x_1^2$

$=22/9$

Problem:-01(Beale's Method)

The operations research team of the ABC company has come up with the mathematical data(daily basis) needed for two product which the firm manufactures. Its also has determined that this is a non-linear programming problem, having linear constraints and objective function which is the sum of a Linear and a quadratic form. The pertinent gathered by the OR team are:

Max $Z=12x+21y+2xy-2x^2-2y^2$

Subject to constrains

i) $8-y\geq 0$

ii) $10-x-y\geq 0$ and $x,y\geq 0$

Find the maximum contribution and number of units that can be expected for these two products which are a part of the firm's total output.(x and y represent the number of units of the two products.)

Solution

Let us rewrite the given QPP as follows

Max $Z=12x+21y+2xy-2x^2-2y^2$

Subject to constraints

$y\leq 8$

$x+y\leq 10$

and $x,y\geq 0$

The standard form of QPP is

Max $Z=12x+21y+2xy-2x^2-2y^2$

Subject to constraints

$y+s_1=8$

$x+y+s_2 = 10$

and $x,y,s_1,s_2\geq 0$

Initial basic feasible solution is

$X_B=(x,y)=(8,2)$

$X_N=(s_1,s_2)=(0,0)$

Let us write the constraints in terms of non basic variables.

$y=8-s_1$

$x=10-y-s_1-s_2=10-8-s_1-s_2=2-s_1-s_2$

Let us write the objective function in terms of none basic variable.

$Z=12(2-s_1-s_2)+21(8-s_1)+2(2-s_1-s_2)-(2-s_1-s_2)^2-(8-s_1)^2$

Z=244-53s_1-28s_2+2 s_1s_2+2s_1^2-2(2 s_1-s_2)^2-2(8- s_1)^2

Differentiate Z partially with respect to non basic variables.

$$\frac{\partial Z}{\partial s_1} = -53 + 2s_2 + 4(2 - s_1 - s_2) + 4(8 - s_1)$$

$$\left.\frac{\partial Z}{\partial s_1}\right|_{\substack{s_1=0 \\ s_2=0}} = -13 = \alpha_1$$

i.e Z decreases when s_1 increases

$$\frac{\partial Z}{\partial s_2} = -28 + 2s_1 + 4(2 - s_1 - s_2);$$

$$\left.\frac{\partial Z}{\partial s_2}\right|_{\substack{s_1=0 \\ s_2=0}} = -20 = \alpha_2$$

i.e z decreases when s_1 increases

Since both the partial derivatives are negative, the current solution is optimum.

Thus the optimum solution x=2, y=8 with max Z=88

Hence in order to have a maximum contribution of Rs88, the ABC company must expect 2 and 8 units of the two products respectively.

Problems on Maxima or Minima

Problem:-01

A trader receive x units of an item at the beginning of each month. The cost of carrying x units per month is given by

$$c(x) = \frac{c_1 x^2}{2n} + \frac{c_2(20n - x)^2}{2n}$$

Where c_1= cost per day of carrying a unit of item in stock(Rs10)

C_2=cost per day of shortage of a unit of item (Rs150)

n= number of units of item to be supplied per day(30)

determine the order quantity x that minimize the cost of inventory

Solution :-

Given $c(x) = \dfrac{c_1 x^2}{2n} + \dfrac{c_2(20n - x)^2}{2n}$ -------(1)

The necessary condition for a function to have either minimum or maximum value at a point is that its first derivative should be zero .

Differentiate equation (1) with respect to x and assume it as zero

$$\frac{d}{dx}c(x) = \frac{2c_1x}{2n} + \frac{2c_2(20n-x)(0-1)}{2n}$$

$$\frac{d}{dx}c(x) = \frac{c_1x}{n} - \frac{c_2(20n-x)}{n} \quad \text{--------(2)}$$

$$0 = \frac{c_1x}{n} - \frac{c_2(20n-x)}{n}$$

$$c_1x - c_2(20n-x) = 0$$

Here c_1=10, c_2=150 and n=30.

$$10x - 150(600-x) = 0$$

$$10x - 150(600-x) = 0$$

x=(600x150)/160=562.5

At this point the function f(x) may have optimum

Thus the nature of the extreme point given by the second derivative.

Differentiate equation (2) with respect to x, we have

$$\frac{d^2c(x)}{dx^2} = \frac{c_1}{n} + \frac{c_2}{n} = 10/30 + 150/30 = 1/3 + 5 = 16/3 > 0$$

The value of the second derivative is positive, therefore f(x) has a local minimum at x=562.5

To find the global minimum

We have to select two extreme values before and after the point x=562.5

say x=0 and infinity.

Substituting the value of x in c(x),we get

C(x=0)=Rs900000

C(562.5)=Rs56249.37;

$$\lim_{x \to \infty} c(x) = \infty$$

It follows that, a global minimum for c(x) occurs at x=562.5

Problem :-02

A firm has a total revenue function, $R=20x-2x^2$, and a total cost function $c=x^2-4x+20$, where x represents the quantity. Find the revenue maximizing output level and the corresponding value of profit, price and total revenue.

Solution:-

Given $R(x)=20x-2x^2$ ----(1) and $C(x)= x^2-4x+20$------(2)

The necessary condition for a revenue function R(x) to have maximum value at a point is

$$\frac{dR}{dx}=0 \text{ and } \frac{d^2R}{dx^2}<0$$

Since $R(x)=20x-2x^2$

Differentiate (1) with respect to x and equivalent it to zero, we have

$$\frac{dR}{dx}=20-4x \text{ -------(3)}$$

=> $20-4x=0$

=> $x=5$.

Also differentiate (3) with respect to x, we get

$$\frac{d^2R}{dx^2}=-4(<0)$$

Hence the revenue will be maximum at an output level at x=5.

The profit function is given by

Profit=Revenue-Cost

$P(x)=R(x)-C(x)=(20x-2x^2)-(x^2-4x+20)=24x-3x^2-20$

Thus, the total profit at x=5 will be $P(5)=24(5)-3(5)^2-20=25$.

The price of a product at x=5 is

$p(x)=R(x)/x=(20x-2x^2)/x=20-2x$

$p(5)=20-2(5)=10$

The maximum revenue at x=5 is $R(5)=20(5)-2(5)^2=50$

Problem:-04

Obtain necessary conditions for the optimum solution of the following problem

Minimize $f(x_1,x_2)= 3e^{2x_1} + 2e^{x_2+5}$

Subject to constraints

$g(x_1,x_2)= x_1+x_2-7=0$

and $x_1,x_2\geq 0$

solution:-

Let us write the Lagrange's function

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$$

$$L(x_1, x_2, \lambda) = 3e^{2x_1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

The necessary conditions for the minimum of $f(x_1,x_2)$ are given by

$$\frac{\partial L}{\partial x_1} = 0$$

$$=> 6e^{2x_1+1} - \lambda = 0$$

$$=> \lambda = 6e^{2x_1+1} \text{-------(1)}$$

$$\frac{\partial L}{\partial x_2} = 0$$

$$=> 2e^{x_2+5} - \lambda = 0$$

$$=> \lambda = 2e^{x_2+5} \text{---------(2)}$$

From (1)and (2), we get

$$=> 2e^{x_2+5} = 6e^{2x_1+1}$$

$$=> \log 2e^{x_2+5} = \log 6e^{2x_1+1}$$

$$=> \log 2 + \log e^{x_2+5} = \log 6 + \log e^{2x_1+1}$$

$$=> \log 2 + x_2 + 5 = \log 6 + 2x_1 + 1$$

$$=> 2x_1 - x_2 = 5 + \log 2 - \log 6 - 1$$

$$=> 2x_1 - x_2 = 4 + \log 2/6$$

$$=> 2x_1 - x_2 = 4 + \log 1/3$$

$$=> 2x_1 - x_2 = 4 + \log 1 - \log 3$$

$=> 2x_1 - x_2 = 4 - \log 3$-------(3)

$$\frac{\partial L}{\partial \lambda} = 0$$

$=> -(x_1 + x_2 - 7) = 0$

$x_1 + x_2 = 7$ --------(4)

Add (3) and (4), we get

$3x_1=11-\log3$

$=>x_1=1/3(11-\log3)$

Substitute the value of x_1 in (4), we get

$1/3(11-\log3) + + x_2 = 7$

$x_2=7-(1/3)(11-\log3)$

Hence the necessary conditions for the optimum solution is

$x_1=1/3(11-\log3)$ and

$x_2=7-(1/3)(11-\log3)$

To find $\lambda = ?$

Sub the value of x_2 in (2), we have

$(2) => \lambda = 2e^{x_2+5}$

$=> \lambda = 2e^{7-(1/3)(11-\log 3)+5}$

$=> \lambda = 2e^{12-(1/3)(11-\log 3)}$

Problem:05

Solve the following problem by using the method of Lagrange's multipliers.

Minimize Z= $x_1^{\ 2} + x_2^{\ 2} + x_3^2$

Subject to constraints

i) $x_1 + x_2 + 3x_3 = 2$

ii) $5x_1 + 2x_2 + x_3 = 5$

and $x_1,x_2 \geq 0$

Solution:-

The Lagrange's function is

$$L(x,\lambda) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + 3x_3 - 2) - \lambda_2(5x_1 + 2x_2 + x_3 - 5)$$

The necessary conditions for the minimum of Z gives

$$\frac{\partial L}{\partial x_1} = 0$$

$$=> 2x_1 - \lambda_1 - 5\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0$$

$$=> 2x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 0$$

$$=> 2x_3 - 3\lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0$$

$$=> -(x_1 + x_2 + 3x_3 - 2) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0$$

$$=> -(5x_1 + 2x_2 + x_3 - 5) = 0$$

The solution of these simultaneous equations gives

$X=(x_1,x_2,x_3)=(37/46,16/46,13/46)$

$$\lambda = (\lambda_1, \lambda_2) = (2/23, 7/23)$$

and Min Z=193/250

To see that this solution corresponds to the minimum of Z apply the sufficient condition by having a matrix;

$$\mathbf{D} = \left[\begin{array}{ccc|cc} 2 & 0 & 0 & 1 & 5 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 2 & 3 & 1 \\ \hline 1 & 1 & 3 & 0 & 0 \\ 5 & 2 & 1 & 0 & 0 \end{array}\right]$$

Since m-2,n=3 so n-m=1 and 2n+m=5, only one minor of D of order 5 needs to be evaluated and it must have a positive sign; $(-1)^m=(-1)^2=1$. Since

$$|D| = 460 > 0$$

The extreme point X=(x_1,x_2,x_3) corresponds to the minimum of Z.

Problem:-06

Use the method of Lagrange's multipliers to solve the following NLP problem. Does the solution maximize or minimize the objective function?

Optimize Z= $2x_1^{\ 2} + x_2^{\ 2} + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

Subject to constrain

$$g(x) = x_1 + x_2 + x_3 = 20$$

and $x_1,x_2,x_3 \geq 0$

Solution

The Lagrange's function can be formulated as

$$L(x,,\lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda_1(x_1 + x_2 + x_3 - 20)$$

The necessary conditions for maximum or minimum are

$$\frac{\partial L}{\partial x_1} = 0$$

$$4x_1 + 10 - \lambda = 0$$

$x_1 = (\lambda - 10)/4$ --------(1)

$$\frac{\partial L}{\partial x_2} = 0$$

$$2x_2 + 8 - \lambda = 0$$

$x_2 = (\lambda - 8)/2$ -----(2)

$$\frac{\partial L}{\partial x_3} = 0$$

$$6x_3 + 6 - \lambda = 0$$

$x_3 = (\lambda - 6)/6$ ----(3)

$$\frac{\partial L}{\partial \lambda} = 0$$

$$-(x_1 + x_2 + x_3 - 20) = 0$$

$$x_1 + x_2 + x_3 = 20 \text{------}(4)$$

Putting the values of (1) , (2) and (3) in (4), we get the value of λ,

(4)=> $(\lambda - 10)/4 + (\lambda - 8)/2 + (\lambda - 6)/6 = 20$

$$[3(\lambda - 10) + 6(\lambda - 8) + 2(\lambda - 6)]/12 = 20$$

$$[3\lambda - 30 + 6\lambda - 48 - 12 + 2\lambda]/12 = 20$$

$$11\lambda - 90 = 240$$

$$11\lambda = 330$$

=> λ=30.

Substituting the value of λ in the other three equations,, we get an extreme point

$$x_1 = (\lambda - 10)/4 = (30 - 10)/4 = 20/4 = 5$$

$$x_2 = (\lambda - 8)/2 = (30 - 8)/2 = 22/2 = 11$$

$$x_3 = (30 - 6)/6 = 24/6 = 4$$

To prove the sufficient condition whether the extreme point solution gives maximum or minimum value of the objective function we evaluate (n-1) principle minors as follows;

$$\Delta_3 = \begin{vmatrix} 0 & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} \\ \dfrac{\partial g}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 g}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} - \lambda \dfrac{\partial^2 g}{\partial x_1\, \partial x_2} \\ \dfrac{\partial g}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2\, \partial x_1} - \lambda \dfrac{\partial^2 g}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 g}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = 48$$

Since sign of $\Delta_3 \, and \, \Delta_4$ are alternative,

Therefore extreme point x_1, x_2, x_3=5,11,4 is a local maximum.

At this point the value of objective function is Z=281.

Problem:-07

Find the optimum value of the objective function when subject to the following three sets of constraints separately.

Maximize Z= $10x_1-x_1^2+10x_2-x^2$

Subject to constrain

(a)

$x_1+x_2\leq 14$

$-x_1+x_2\leq 6$

$x_1,x_2\geq 0$

(b)

$x_1+x_2\leq 8$

$-x_1+x_2\leq 5$

$x_1,x_2\geq 0$

(c)

$x_1+x_2\leq 9$

$x_1-x_2\leq 6$

$x_1,x_2\geq 0$

solution

(a) Here the constraints are;

Let $g_1(x)= x_1+x_2+s_1^2-14=0$

Let $g_2(x)= -x_1+x_2+s_2^2-6=0$

The Lagrange's function is

$$L(x,\lambda,s) = 10x_1 - x_1^2 + 10x_2 + x_2^2 - \lambda_1(x_1 + x_2 + s_1^2 - 14) - \lambda_2(-x_1 + x_2 + s_2^2 - 6)$$

Kuhn-tucker necessary conditions for a maximize problem are

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1^2 - 14) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(x_1 + x_2 + s_2{}^2 - 6) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0$$

$$\frac{\partial L}{\partial s_2} = -2\lambda_1 s_2 = 0$$

The unconstrained solution (ie. $\lambda_1=\lambda_2=0$) obtained by solving the first equation is $x_1=5$, $x_2=5$, $s_1{}^2=4$, $s_2{}^2=6$ and max Z=50

Since both $s_1{}^2$ and $s_2{}^2$ are positive the solution is feasible. As the solution so obtained is unconstrained, so to fine whether or not the solution is maximum we test the hessian matrix for the given objective function as;

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 Z}{\partial x_1^2} & \frac{\partial^2 Z}{\partial x_1\, \partial x_2} \\ \frac{\partial^2 Z}{\partial x_2\, \partial x_1} & \frac{\partial^2 Z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

and $\det A_1 = \left| \frac{\partial^2 Z}{\partial x_1^2} \right| = -2; \quad \det A_2 = |H| = 4$

Since signs of the principle minors of H are alternating, matrix H is negative definite and the point x=(4,4) gives the local maximum of the objective function Z.

(b) Here the constraints are

Let $g_1(x)= x_1+x_2+s_1{}^2-8=0$

Let $g_2(x)= -x_1+x_2+s_2{}^2-5=0$

The Lagrange's function is

$$L(x,\lambda,s) = 10x_1 - x_1^2 + 10x_2 + x_2^2 - \lambda_1(x_1 + x_2 + s_1^2 - 8) - \lambda_2(-x_1 + x_2 + s_2{}^2 - 5)$$

Kuhn-tucker necessary conditions for a maximize problem are

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1{}^2 - 8) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(x_1 + x_2 + s_2{}^2 - 5) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0$$

$$\frac{\partial L}{\partial s_2} = -2\lambda_1 s_2 = 0$$

The unconstrained solution (ie. $\lambda_1=\lambda_2=0$) obtained by solving the first equation is $x_1=5$, $x_2=5$, $s_1{}^2=-2$, $s_2{}^2=5$ and max Z=50

Since $s_1{}^2=-2$ this solution is infeasible. Solving again these equation for $s_1=\lambda_1=0$, we get $x_1=4$, $x_2=4$, $s_2{}^2=5$, $\lambda_1=2$ and Max Z=48.

This solution satisfies both the constraints and the conditions $s_1\lambda_1= s_2\lambda_2=0$ are also satisfied, therefore the point X=(4,4) gives the maximum of objective function Z.

(c) Here the constraints are

Let $g_1(x)= x_1+x_2+s_1{}^2-9=0$

Let $g_2(x)= -x_1+x_2+s_2{}^2+6=0$

The Lagrange's function is

$$L(x,\lambda,s) = 10x_1 - x_1^2 + 10x_2 + x_2^2 - \lambda_1(x_1 + x_2 + s_1^2 - 9) - \lambda_2(-x_1 + x_2 + s_2{}^2 + 6)$$

Kuhn-tucker necessary conditions for a maximize problem are

$$\frac{\partial L}{\partial x_1} = 10 - 2x_1 - \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 10 - 2x_2 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + s_1{}^2 - 9) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = -(x_1 + x_2 + s_2{}^2 + 6) = 0$$

$$\frac{\partial L}{\partial s_1} = -2\lambda_1 s_1 = 0$$

$$\frac{\partial L}{\partial s_2} = -2\lambda_1 s_2 = 0$$

The unconstrained solution (ie. $\lambda_1=\lambda_2=0$) obtained by solving the first equation is $x_1=8$, $x_2=2$, $s_1^2=-1$, $s_2^2=-6$ and max Z=50 since both s_1^2 and s_2^2 are negative, the solution is infeasible.

Solving again these four equation for $s_1=\lambda_1=0$(violating second condition) we get $x_1=2$, $x_2=8$, $s_1^2=-1$, $\lambda_2=6$ and Max Z=32 this solution is also infeasible as s_1^2 is negative.

Solving again these four equation for $s_1=s_2=0$(ie. $\lambda_1=\lambda_2=0$) we get $x_1=7.5$, $x_2=1.5$, $\lambda_1=1$ $\lambda_2=6$and Max Z=31.50. since this solution does not violate any of the condition,

Therefore, the point x=(7.5,1.5) gives the maximum of the objective function Z.

Problem :-08

Determine x_1 and x_2 so as to

Maximize $Z=12x_1+21x_2+2x_1x_2-2x_1^2-2x_2^2$

Subject to constrains

(i) $x_2 \le 8$

(ii)$x_1+x_2 \le 8$

and $x_1,x_2 \ge 0$

Solution

Here $f(x_1, x_2)= 12x_1+21x_2+2x_1x_2-2x_1^2-2x_2^2$

Let $g_1(x_1, x_2)= x_2-8=0$

Let $g_2(x_1, x_2)=x_1+ x_2-10=0$

The Lagrange's function is

$$L(x,s,\lambda) = f(x) - \lambda_1[g_1(x) + s_1^2] - \lambda_2[g_2(x) + s_2^2]$$

Kuhn-tucker necessary conditions can be stated as

(i) $\dfrac{\partial f}{\partial x_j} = \sum_{i=1}^{2} \lambda_i \dfrac{\partial g_1}{\partial x_j} = 0;\ j = 1,2$

(ii) $\lambda_i g_1(x) = 0;\ i = 1,2$

$12+2x_2-4x_1-\lambda_2=0$

$\lambda_1(x_2-8)=0$

$21+2x_1-4x_2- \lambda_{1-} \lambda_2=0$

$\lambda_2(x_1+ x_2-10)=0$

(iii) $g_1(x) \le 0$

$x_2-8\le0$

$x_1+ x_2-10\le0$

(iv) $\lambda_i \ge 0; i = 1,2$

These may arise four case

Case 1

If $\lambda_1=0$ $\lambda_2=0$, then from condition(i) we have

$12+2x_2-4x_1=0$ and $21+2x_1-4x_2- \lambda_1 - \lambda_2=0$

Solving these equations we get $x_1=15/2$, $x_2=9$ however, this solution violates condition (iii) and therefore it may be discarded.

Case 2

$\lambda_1\neq0$ $\lambda_2\neq0$, then from condition(ii) we have

$x_2-8=0$ or $x_2=8$and $x_1+ x_2-10=0$ or $x_1=2$

substituting these values in condition (i) we get $\lambda_1=-27$ $\lambda_2=20$ however this solution violates the condition (iv) and therefore may be discarded.

Case 3

$\lambda_1\neq0$ $\lambda_2=0$, then from condition(ii)and (i) we have

(a) $x_1+x_2=10$

(b)$2x_2-4x_1=-12$

(c)$2x_1-4x_2=-12+ \lambda_1$

Solving these equations we get $x_1=2$, $x_2=8$ and $\lambda_1=-16$. However this solution violates the condition(iv) and therefore maybe discarded.

Case 4

$\lambda_1=0$ $\lambda_2\neq0$, then from condition(i)and (ii) we have

(a) $2x_1-4x_2=-12$

(b)$2x_1-4x_1=-21+ \lambda_1$

(c)$x_1+x_2=10$

Solving these equations we get $x_1=17/4$, $x_2=23/4$and $\lambda_2=13/4$. This solution does not violate any of the kuhn-tucker condition and therefore must be accepted.

Hence the optimum solution of the given problem is $x_1=17/4$, $x_2=23/4$, $\lambda_1=0$and $\lambda_2=13/4$ and max Z=1734/16.